Some Sequences Associated with the Golden Ratio

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A number of people have considered the arithmetical, combinatorial, geometrical, and other properties of sequences of the form $([n\alpha] : n \geq 1)$, where α is a positive irrational number and [] denotes the greatest integer function. (See, e.g., [1–16] and the references contained in those papers, especially [8] and [16].)

There are several other sequences which may be naturally associated with the sequence $([n\alpha] : n \geq 1)$. They are the *difference sequence*

$$f_\alpha(n) = [(n+1)\alpha] - [n\alpha] - [\alpha]$$

(the difference sequence is "normalized" by subtracting $[\alpha]$ so that its values are 0 and 1), the *characteristic function*

$$g_\alpha(n) \quad (g_\alpha(n) = 1 \text{ if } n = [k\alpha] \text{ for some } k, \text{ and } g_\alpha(n) = 0 \text{ otherwise}),$$

and the *hit sequence*

$$h_\alpha(n),$$

where $h_\alpha(n)$ is the number of different values of k such that $[k\alpha] = n$.

We use the notation

$$f_\alpha = (f_\alpha(n) : n \geq 1), \quad g_\alpha = (g_\alpha(n) : n \geq 1), \quad h_\alpha = (h_\alpha(n) : n \geq 1).$$

Note that $f_\alpha = f_{\alpha+k}$ for any integer $k \geq 1$. In particular, $f_\alpha = f_{\alpha-1}$ if $\alpha > 1$.

Special properties of these sequences in the case where α equals τ, the golden mean, $\tau = (1+\sqrt{5})/2$, are considered in [5, 12, 14, 16]. For example, the following is observed in [12]. Let $u_n = [n\tau], n \geq 1$, and let F_k denote the kth Fibonacci number. Given k, let $r = F_{2k}, s = F_{2k+1}, t = F_{2k+2}$. Then

$$u_r = s, \quad u_{2r} = 2s, \quad u_{3r} = 3s, \ldots, u_{t-2}r = (t-2)s;$$

thus, the sequence $([n\tau])$ contains the $(t-2)$-term arithmetic progression $(s, 2s, 3s, \ldots, (t-2)s)$.

It was shown in [16], using a theorem of A. A. Markov [11] (which describes the sequence f_α (for any α) explicitly in terms of the simple continued fraction expansion of α), that the difference sequence

f_τ has a certain "substitution property." We give a simple proof of this below (Theorem 2) without using Markov's theorem. We also make several observations concerning the three sequences f_τ, g_τ, and h_τ.

Theorem 1. *The golden mean τ is the smallest positive irrational real number α such that $f_\alpha = g_\alpha = h_\alpha$. In fact, $f_\alpha = g_\alpha = h_\alpha$ exactly when $\alpha^2 = k\alpha + 1$, where $k = [\alpha] \geq 1$.*

Proof. It follows directly from the definitions (we omit the details) that if α is irrational and $\alpha > 1$, then $h_\alpha = g_\alpha = f_{1/\alpha}$. (The fact that $g_\alpha = f_{1/\alpha}$ is mentioned in [8]. It is straightforward to show that

$$g_\alpha(n) = 1 \Rightarrow f_{1/\alpha}(n) = 1 \quad \text{and} \quad g_\alpha(n) = 0 \Rightarrow f_{1/\alpha}(n) = 0.)$$

Also, if α is irrational and $\alpha > 0$, then

$$h_\alpha(n) = f_{1/\alpha}(n) + [1/\alpha] \quad \text{for all } n \geq 1.$$

Thus, if α is irrational and $f_\alpha = g_\alpha = f_\alpha$, then $\alpha > 1$ (otherwise, g_α is identically equal to 1, and f_α is not) and

$$f_{\alpha-[\alpha]}(n) = f_\alpha(n) = g_\alpha(n) = f_{1/\alpha}(n) \quad \text{for all } n \geq 1.$$

Since the sequence f_β determines β if $\beta < 1$, this gives $\alpha - [\alpha] = 1/\alpha$, and the result follows. □

Definition 1. For any finite or infinite sequence w consisting of 0's and 1's, let $\bar{w}$ be the sequence obtained from w by replacing each 0 in w by 1, and each 1 in w by 10. For example, $\overline{10110} = 10110101$. (Compare "Fibonacci strings" [10, p. 85].)

Note that $\overline{uv} = \bar{u} \cdot \bar{v}$, and that $\bar{u} = \bar{v} \Rightarrow u = v$ by induction on the length of v.

Theorem 2. *The sequences f_τ and $\overline{f_\tau}$ are identical.*

Proof. First, we show that if $0 < \alpha < 1$, then $\overline{f_\alpha} = g_{1+\alpha}$. Let $L(w)$ denote the *length* of the finite sequence w, so that if $w = f_\alpha(1) f_\alpha(2) \cdots f_\alpha(k)$, then

$$L(\bar{w}) = k + f_\alpha(1) + \cdots + f_\alpha(k) = k + [(k+1)\alpha].$$

Thus,

$$\begin{aligned}[\overline{f_\alpha}(n) = 1] &\Leftrightarrow [n = L(\bar{w}) + 1 \text{ for some initial segment } w \text{ of } f_\alpha] \\ &\Leftrightarrow [n = [(k+1)(1+\alpha)] \text{ for some } k \geq 0] \Leftrightarrow [g_{1+\alpha}(n) = 1].\end{aligned}$$

Therefore, $\overline{f_\tau} = \overline{f_{\tau-1}} = g_\tau = f_{1/\tau} = f_{\tau-1} = f_\tau$. □

Corollary 1. *The sequence f_τ can be generated by starting with $w = 1$ and repeatedly replacing w by $\bar{w}$.*

Proof. If we define $E_1 = 1$ and $E_{k+1} = \overline{E_k}$, then, since $\bar{1} = 10$ begins with a 1, it follows that, for each k,

E_k is an initial segment of E_{k+1}. By Theorem 2 and induction, each E_k is an initial segment of f_τ. Thus,

$$E_1 = 1, \quad E_2 = \overline{E_1} = 10, \quad E_3 = \overline{E_2} = 101, \quad E_4 = \overline{E_3} = 10110,$$
$$E_5 = \overline{E_4} = 10110101, \quad \text{etc.},$$

are all initial segments of f_τ. (These blocks naturally have lengths $1, 2, 3, 5, 8, \ldots$.) □

Corollary 2. *For each $i \geq 1$, let x_i denote the number of 1's in the sequence f_τ which lie between the ith and $(i+1)$st 0's. Thus,*

$$f_\tau = 101101011011010110101101101011011\cdots,$$
$$(x_n) = \quad 2\ 1\ \ 2\ \ 2\ 1\ \ 2\ 1\ \ 2\ \ 2\ 1\ \ 2\ \ 2\cdots.$$

Then the sequences $(x_n - 1)$ and f_τ are identical.

Proof. If we start with the sequence (x_n) and replace each 1 by 10 and each 2 by 101, we obtain the sequence f_τ. Since $\bar{\bar{0}} = 10$ and $\bar{\bar{1}} = 101$, this shows that $\overline{\overline{(x_n - 1)}} = f_\tau = \overline{\overline{f_\tau}}$. Therefore, $\overline{(x_n - 1)} = \overline{f_\tau}$, and, finally, $(x_n - 1) = f_\tau$. □

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