A Characterization of the Quadratic Irrationals

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Abstract

Let \( \alpha \) be a positive irrational real number, and let \( f_\alpha(n) = \left[ (n+1)\alpha \right] - \left[ n\alpha \right] - \lfloor \alpha \rfloor, n \geq 1 \), where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \). It is shown that the sequence \( f_\alpha \) has a certain ‘substitution property’ if and only if \( \alpha \) is the root of a quadratic equation over the rationals.

1 Introduction

The astronomer J. Bernoulli [1] considered the sequence \( ((n+1)\alpha + 1/2 - n\alpha + 1/2 : n \geq 1) \), for positive irrational numbers \( \alpha \), and gave (without proof) an explicit description of the terms of this sequence, based on the simple continued fraction expansion of \( \alpha \).

A. A. Markov [5] proved the validity of Bernoulli’s description, and did this by first describing the terms of the sequence \( f_\alpha = (f_\alpha(n) : n \geq 1) \), where \( f_\alpha(n) = \left[ (n+1)\alpha \right] - \left[ n\alpha \right] - \lfloor \alpha \rfloor \). A beautiful exposition (about 2 pages long) of Markov’s proof is given by B. A. Venkov [8].

K. B. Stolarsky [7] gave a different description of the sequence \( f_\alpha \), for certain values of \( \alpha \). A. S. Fraenkel, M. Mushkin, and U. Tassa [3] gave a very short and polished proof which extended Stolarsky’s result to all positive \( \alpha \), including rational values. Both [7] and [3] contain extensive lists of references.

Stolarsky gave two proofs of his result. In his second proof, he used Markov’s theorem to show that if \( \alpha = \left[ 0, k, k, \ldots \right] = 1/k + 1/k + \ldots \), then the sequence \( f_\alpha \), which is a sequence of 0’s and 1’s, is invariant under the substitution \( 0 \rightarrow B_1 = 0^{k-1}1, 1 \rightarrow B_2 = 0^{k-1}10 \). To say that \( f_\alpha \) is invariant under this substitution means that if each 0 in \( f_\alpha \) is replaced by the block \( B_1 \), and each 1 is replaced by \( B_2 \), then the resulting sequence is identical with \( f_\alpha \). Here \( 0^{k-1}1 \) indicates a block of \( k-1 \) consecutive 0’s, and if \( k = 1 \) then \( 0^{k-1} \) is the empty block. (If \( k = 3 \), the substitution is \( 0 \rightarrow 001, 1 \rightarrow 0010 \).) In this note we give a simple proof, which does not use Markov’s theorem, of a generalization of Stolarsky’s result. Our main results are based on the fact that if \( \alpha \) is any quadratic irrational, then the simple continued fraction for \( \alpha \) is periodic.

Throughout, we used the standard notation \( [a_0, a_1, a_2, \ldots] \) for the simple continued fraction \( a_0 + 1/(a_1 + 1/(a_2 + \cdots)) \).

We show that if \( \alpha = [0, a_1, \ldots, a_m] = [0, a_1, \ldots, a_m, a_1, \ldots, a_m, \ldots] = [0, a_1, \ldots, a_m + \alpha] \) then there are blocks \( B_1 \) and \( B_2 \) (not both of length 1) such that \( f_\alpha \) is invariant under the substitution \( 0 \rightarrow B_1, 1 \rightarrow B_2 \).

We also show that such blocks \( B_1 \) and \( B_2 \) cannot be found for all quadratic irrationals \( \alpha \).
However, we show that it is true that for every quadratic irrational $\alpha$, $f_\alpha$ is invariant under a substitution of the following kind. There always exist blocks $s, t, C_1, C_2$ of $0$'s and $1$'s (where $C_1$ is longer than $s$ or $C_2$ is longer than $t$) such that $f_\alpha$ may be written as a sequence of $s$'s and $t$'s, $C_1$ and $C_2$ may be written as blocks of $s$'s and $t$'s, and $f_\alpha$ is invariant under the substitution $s \mapsto C_1, t \mapsto C_2$.

Finally, we show that if $f_\alpha$ is invariant under a substitution in the above sense, then $\alpha$ is a quadratic irrational. Thus the 'substitution property' of $f_\alpha$ characterizes quadratics among the irrationals.

## 2 Results

We will make use of the sequence $g_\alpha = (g_\alpha(n) : n \geq 1)$, the characteristic function of the sequence $([n\alpha] : n \geq 1)$, where $g_\alpha(n) = 1$ if $n = [k\alpha]$ for some $k$, and $g_\alpha(n) = 0$ otherwise. We will also use the fact (from the definition of $f_\alpha$) that if $j$ is any integer with $0 \leq j < \alpha$, then $f_\alpha = f_\alpha-j$.

It will be convenient to use the notation $1/b = [0, b], 1/(a + 1/b) = [0, a, b]$, etc., for positive real numbers $a, b$.

We begin with a fact which is mentioned by Fraenkel, Mushkin, and Tassa [3]:

**Lemma.** For any irrational $\alpha > 1$, $g_\alpha = f_{1/\alpha}$.

**Proof.** It is straightforward to show from the definitions of $g_\alpha$ and $f_{1/\alpha}$ that $g_\alpha(n) = 1 \Rightarrow f_{1/\alpha}(n) = 1$ and $g_\alpha(n) = 0 \Rightarrow f_{1/\alpha}(n) = 0$.

**Definition 1.** Let $k \geq 1$ be fixed, and let $w$ be any block of $0$'s and $1$'s or any sequence of $0$'s or $1$'s. Then $h_k(w)$ is obtained from $w$ by applying the substitution $0 \mapsto 0^{k-1}1$, $1 \mapsto 0^{k-1}10$, where $0^{k-1}$ is a block of $k-1$ consecutive $0$'s. That is, $h_k(w)$ is obtained from $w$ by replacing each $0$ in $w$ by $0^{k-1}1$, and each $1$ by $0^{k-1}10$. If $k = 1$ the substitution is $0 \mapsto 1, 1 \mapsto 10$.

**Lemma.** Let $k \geq 1$ and $\alpha$ be given, where $\alpha$ is irrational and $0 < \alpha < 1$. Then $h_k(f_\alpha) = g_{k+\alpha} = f_{1/(k+\alpha)}$.

**Proof.** By definition,

$$f_\alpha = f_\alpha(1)f_\alpha(2) \cdots f_\alpha(j) \cdots,$$

where

$$f_\alpha(j) = \lfloor (j + 1)\alpha \rfloor - \lfloor j\alpha \rfloor, \quad j \geq 1,$$

and

$$h_k(\alpha) = D_1D_2 \cdots D_qD_{q+1} \cdots,$$

where

$$D_j = h_k(f_\alpha(j)), \quad j \geq 1.$$

Note that each block $D_j$ contains exactly one “$1$”, which is in the $k$th position, and has length either $k$ or $k + 1$.

Consider $h_k(f_\alpha)$ now as a sequence of $0$'s and $1$'s, and let $n$ be the position in this sequence of the $(q + 1)$st “$1$”, that is, the $1$ in the block $D_{q+1}$. Then

$$n = L(D_1D_2 \cdots D_q) + k,$$
Proof. Let $E_1 = f_{a_1}(1)$ and for $k \geq 1$ let $E_{k+1} = H(E_k)$. Then by the Theorem and induction, each $E_k$ is an initial segment of $f_{a_0}$.

**Theorem 2.** Let $\beta > 0$ be any quadratic irrational. Since $f_\beta = f_{\beta-|\beta|}$, assume without loss of generality that $0 < \beta < 1$, so that (for suitable $a_i, b_i$) $\beta = \left[0, b_1, \ldots, b_q, \overline{a_1, \ldots, a_m}\right]$. Let

$$s = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q}(0)$$

$$t = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q}(1),$$

where $L(D_1D_2 \cdots D_q)$ denotes the length of $D_1D_2 \cdots D_q$.

Since the block $D_j$ has length $k$ if $f_{a_j}(j) = 0$ and has length $k + 1$ if $f_{a_j}(j) = 1$, it follows that

$$L(D_1D_2 \cdots D_q) = qk + f_{a_1}(1) + \cdots + f_{a_q}(q).$$

Since $f_{a_0}(j) = [(j + 1)\alpha] - [j\alpha]$, and $[\alpha] = 0$, the sum telescopes to

$$L(D_1D_2 \cdots D_q) = qk + [(q + 1)\alpha].$$

Thus $n$, the position of the $(q + 1)^{\text{th}}$ “1” in the sequence $h_k(f_{a_0})$ satisfies

$$n = qk + [(q + 1)\alpha] + k = [(q + 1)(k + \alpha)].$$

Thus $[h_k(f_{a_0})(n) = 1] \Leftrightarrow [n = [(q + 1)(k + \alpha)]$ for some $q \geq 0] \Leftrightarrow [g_{k+\alpha}(n) = 1]$. That is, $h_k(f_{a_0}) = g_{k+\alpha}$.

Using Lemma 2, this gives

$$h_k(f_{a_0}) = g_{k+\alpha} = f_{1/(k+\alpha)}.$$  

**Theorem 1.** Let $\alpha = [0, \overline{a_1, \ldots, a_m}]$. Then $f_{a_0}$ is invariant under the substitution

$$0 \mapsto B_1 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(0)$$

$$1 \mapsto B_2 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(1),$$

where $\circ$ denotes composition.

**Proof.** By Lemma 2, we have $h_{a_0}(f_{a_0}) = f_{1/(a_0+\alpha)} = f_{[0,a_0+\alpha]} ; h_{a_0-1} \circ h_{a_0}(f_{a_0}) = f_{1/(a_0+\alpha)} ; h_{a_0-2} \circ \cdots \circ h_{a_0}(f_{a_0}) = f_\beta$, where $\beta = [0, a_1, a_2, \ldots, a_m+\alpha] = \alpha$.

**Remark.** The blocks $B_1 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(0)$ and $B_2 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(1)$ can be described as follows. Let $S_0 = 0$, $S_1 = 0^{a_1-1}1$, and for $2 \leq j \leq m$, let $S_j = (S_{j-1})^{a_j}\overline{a_j-1}$. Then $B_1 = S_m$, and $B_2 = S_mS_{m-1}$. This can be seen by induction on $m$.

**Corollary.** For any block $w$ of 0’s and 1’s, let $H(w)$ be obtained from $w$ by replacing each 0 by $B_1$, and each 1 by $B_2$. (That is, $H = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}$.) Then $f_{a_0}$ can be generated by starting with $w = f_{a_0}(1)$ and repeatedly replacing $w$ by $H(w)$.

**Proof.** Let $E_1 = f_{a_0}(1)$ and for $k \geq 1$ let $E_{k+1} = H(E_k)$. Then by the Theorem and induction, each $E_k$ is an initial segment of $f_{a_0}$.
and

\[ C_1 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_q} \circ h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(0) \]
\[ C_2 = h_{b_0} \circ h_{b_2} \circ \cdots \circ h_{b_q} \circ h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(1). \]

Then \( f_\beta \), \( C_1, C_2 \), can be written as sequences of \( s \)'s and \( t \)'s and \( f_\beta \) is invariant under the substitution \( s \mapsto C_1, t \mapsto C_2 \).

**Proof.** Let \( \alpha = [0, \alpha_1, \ldots, \alpha_m] \), so that \( \beta = [0, b_1, b_2, \ldots, b_q + \alpha] \). Let \( H_1 = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q}, H_2 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}, B_1 = H_2(0), B_2 = H_2(1) \). Then we have \( s = H_1(0), t = H_1(1) \), and \( C_1 = H_1(B_1), C_2 = H_1(B_2) \), so that \( C_1, C_2 \) can be written as blocks of \( s \)'s and \( t \)'s.

By the proof of Theorem 1, \( H_1(f_\alpha) = f_\beta \), so that \( f_\beta \) can be written as a sequence of \( s \)'s and \( t \)'s, and \( H_2(f_\alpha) = f_\alpha \). Note that \( H_1(f_\alpha) = f_\beta \) is obtained from \( f_\alpha \) by applying the substitution \( 0 \mapsto s, 1 \mapsto t \), and \( H_2(f_\alpha) = f_\alpha \) is obtained by applying the substitution \( 0 \mapsto B_1, 1 \mapsto B_2 \).

Therefore we can transform \( f_\beta \) into \( f_\alpha \) by successively applying the substitutions \([s \mapsto 0, t \mapsto 1], [0 \mapsto B_1, 1 \mapsto B_2], \) and \([B_1 \mapsto C_1, B_2 \mapsto C_2] \), which transform \( f_\beta \) into \( f_\alpha \), \( f_\alpha \) into \( f_\alpha \), and \( f_\alpha \) into \( f_\beta \) respectively. \( \square \)

**Theorem 3.** Let \( \beta = [0, 5, 1, 1, \ldots] \). Then there do not exist non-trivial blocks \( B_1, B_2 \) of 0's and 1's such that \( f_\beta \) is invariant under the substitution \( 0 \mapsto B_1, 1 \mapsto B_2 \).

**Proof.** Let \( \alpha = [0, 1, 1, \ldots] = (\sqrt{5} - 1)/2 \). Then \( f_\alpha(1) = 1 \), and by Theorem 1, \( f_\alpha \) is invariant under \( 0 \mapsto 1, 1 \mapsto 10 \), so \( f_\alpha = 10110 \cdots \). By the proof of Theorem 1, \( h_5(f_\alpha) = f_\beta \), so \( f_\beta = tsttst \cdots \), where \( s = 00001, t = 000010 \). Thus

\[ f_\beta = 00001000001 0000100001 0000100001 0000100001 000010 \cdots \]
\[ = t \quad s \quad t \quad t \quad s \quad t \quad s \quad t \cdots . \]

It was shown by Karhumaki [4] that the sequence \( f_\alpha = 10110 \cdots \) does not contain any 4th power. That is, \( f_\alpha \) contains no non-empty block of the form \( DDDD \). (See also [2] and [6].) Thus also the sequence \( f_\beta = tsttstst \cdots \), when regarded as a sequence of \( s \)'s and \( t \)'s, contains non non-empty block of the form \( DDDD \), where \( D \) is a block of \( s \)'s and \( t \)'s.

Now suppose that \( f_\beta \) is invariant under \( 0 \mapsto B_1, 1 \mapsto B_2 \), and for any block or sequence \( w \) of 0's and 1's, let \( H(w) \) be the result of applying this substitution to \( w \). Note that \( f_\beta = H(f_\beta) = B_1B_1B_1B_1B_2 \cdots \), so that \( B_1 \) is some initial segment of \( f_\beta \).

Our goal is to show that \( B_1 \) can be written as a block of \( s \)'s and \( t \)'s, which form an initial segment of \( f_\beta \), when \( f_\beta \) is regarded as a sequence of \( s \)'s and \( t \)'s. Since \( f_\beta = B_1B_1B_1 \cdots \), this will contradict Karhumaki’s result, and the proof will be finished.

To this end, first note that the block \( B_1 \) must contain at least one “1”, since otherwise, by the Corollary to Theorem 1, \( f_\beta \) would be identically 0. Note also that \( f_\beta \) consists of blocks of either 4 or 5 consecutive 0’s separated by single 1’s. This implies that the block \( B_1 \) ends either with 1 or with 10, since otherwise \( f_\beta = H(f_\beta) = B_1B_1B_1B_1B_2 \cdots \) would contain a block of more than five consecutive 0's. (For example if \( B_1 = C100 \), then since \( C = 00001D \) (which is true since \( B_1 \) is an initial segment of \( f_\beta = 000010 \cdots \)), we would
have
\[ f_\beta = H(f_\beta) = B_1 B_1 \cdots = C100C100 \cdots = C10000001D100\cdots, \]
with too many consecutive 0’s.)

Suppose now that \( B_1 \) fails to be an initial segment of \( s \)'s and \( t \)'s in \( f_\beta \) (when \( f_\beta \) is regarded as a sequence of \( s \)'s and \( t \)'s).

If \( B_1 \) ends in 0, then \( B1 = C0001 \), where \( C0001 \) is an initial segment of \( s \)'s and \( t \)'s in \( f_\beta \) (when \( f_\beta \) is regarded as a sequence of \( s \)'s and \( t \)'s). Note that \( C = 00001D \). Then since \( C0001 \) is an initial segment of \( f_\beta \), we have \( f_\beta = C00001 \cdots \), but also we have
\[ f_\beta = H(f_\beta) = B_1 B_1 \cdots = C00001D00001 \cdots, \]
a contradiction.

If \( B_1 \) ends in 1, then \( B1 = C0001, \) where \( C0001 \) is an initial segment of \( s \)'s and \( t \)'s in \( f_\beta \) (when \( f_\beta \) is regarded as a sequence of \( s \)'s and \( t \)'s). Note again that \( C = 00001D \). Then since \( C0001 \) is an initial segment of \( f_\beta \), we have \( f_\beta = C00001000001 \cdots \), but also we have
\[ f_\beta = H(f_\beta) = B_1 B_1 \cdots = C00001CD00001 \cdots, \]
a contradiction.

Thus we have shown that if \( f_\beta \) is invariant under a substitution \( 0 \mapsto B_1,1 \mapsto B_2 \), then \( B_1 \) can be written as a block of \( s \)'s and \( t \)'s, which form an initial segment of \( f_\beta \), when \( f_\beta \) is regarded as a sequence of \( s \)'s and \( t \)'s. This gives the desired contradiction to Karhumaki’s result, and completes the proof. \( \square \)

**Theorem 4.** Let \( \alpha \) be a positive irrational real number, and let \( f_\alpha \) be written as a sequence on \( s,t \), where \( s,t \) are blocks of 0’s and 1’s. Let \( C_1,C_2 \) be blocks of \( s \)'s and \( t \)'s such that \( f_\alpha \) is invariant under the non-trivial substitution \( s \mapsto C_1, t \mapsto C_2 \). Then \( \alpha \) is a quadratic irrational.

**Proof.** First consider the case where \( 0 < \alpha < 1 \) and \( s = 0, t = 1 \). Suppose that \( C_1 \) contains a 0’s and \( b \) 1’s, and that \( C_2 \) contains \( c \) 0’s and \( d \) 1’s.

For any block \( w \) of 0’s and 1’s, let \( H(w) \) denote the word obtained from \( w \) by replacing each 0 by \( C_1 \) and each 1 by \( C_2 \). Let \( E_1 = H(f_\alpha(1)) \) and for \( p \geq 1 \) let \( E_{p+1} = H(E_p) \). Let \( e_p \) denote the number of 1’s which occur in the block \( E_p \). Since the number of 1’s which occur in the block \( f_\alpha(1)f_\alpha(2) \cdots f_\alpha(n) \) is \( f_\alpha(1) + f_\alpha(2) + \cdots f_\alpha(n) \), which equals \( [(n+1)\alpha] \), we have \( e_p = [(L(E_p) + 1)\alpha] \). Also, \( e_{p+1} = e_p + (L(E_p) - e_p)b \), and \( L(E_{p+1}) = e_p(c + d) + (L(E_p) - e_p)(a + b) \), so that
\[
\frac{e_{p+1}}{L(E_{p+1})} = \frac{e_p + (1 - \frac{e_p}{L(E_p)})b}{\frac{e_p(c + d) + (1 - \frac{e_p}{L(E_p)})b}{L(E_p)}}.
\]
Taking the limit as \( p \to \infty \), we obtain
\[
\alpha = \frac{\alpha + (1 - \alpha)b}{\alpha(c + d) + (1 - \alpha)(a + b)},
\]
so that \( \alpha \) is a quadratic irrational.
The general case can be handled similarly. We omit the details.

References


