A Characterization of the Quadratic Irrationals

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Abstract

Let α be a positive irrational real number, and let $f_\alpha(n) = [(n+1)\alpha] - [n\alpha] - [\alpha]$, $n \geq 1$, where $[x]$ denotes the greatest integer not exceeding x. It is shown that the sequence f_α has a certain 'substitution property' if and only if α is the root of a quadratic equation over the rationals.

1 Introduction

The astronomer J. Bernoulli [1] considered the sequence $([(n+1)\alpha + 1/2] - [n\alpha + 1/2] : n \geq 1)$, for positive irrational numbers α, and gave (without proof) an explicit description of the terms of this sequence, based on the simple continued fraction expansion of α.

A. A. Markov [5] proved the validity of Bernoulli's description, and did this by first describing the terms of the sequence $f_\alpha = (f_\alpha(n) : n \geq 1)$, where $f_\alpha(n) = [(n+1)\alpha] - [n\alpha] - [\alpha]$. A beautiful exposition (about 2 pages long) of Markov's proof is given by B. A. Venkov [8].

K. B. Stolarsky [7] gave a different description of the sequence f_α, for certain values of α. A. S. Fraenkel, M. Mushkin, and U. Tassa [3] gave a very short and polished proof which extended Stolarsky's result to all positive α, including rational values. Both [7] and [3] contain extensive lists of references.

Stolarsky gave two proofs of his result. In his second proof, he used Markov's theorem to show that if $\alpha = [0, k, k, \ldots] = \frac{1}{k+}\frac{1}{k+}\frac{1}{k+}\cdots$, then the sequence f_α, which is a sequence of 0's and 1's, is invariant under the substitution $0 \mapsto B_1 = 0^{k-1}1, 1 \mapsto B_2 = 0^{k-1}10$. To say that f_α is invariant under this substitution means that if each 0 in f_α is replaced by the block B_1, and each 1 is replaced by B_2, then the resulting sequence is identical with f_α. Here 0^{k-1} indicates a block of $k-1$ consecutive 0's, and if $k = 1$ then 0^{k-1} is the empty block. (If $k = 3$, the substitution is $0 \mapsto 001$, $1 \mapsto 0010$.) In this note we give a simple proof, which does not use Markov's theorem, of a generalization of Stolarsky's result. Our main results are based on the fact that if α is any quadratic irrational, then the simple continued fraction for α is periodic.

Throughout, we used the standard notation $[a_0, a_1, a_2, \ldots]$ for the simple continued fraction $a_0 + 1/(a_1 + 1/(a_2 + \cdots))$.

We show that if $\alpha = [0, \overline{a_1, \ldots, a_m}] = [0, a_1, \ldots, a_m, a_1, \ldots, a_m, \ldots] = [0, a_1, \ldots, a_m + \alpha]$ then there are blocks B_1 and B_2 (not both of length 1) such that f_α is invariant under the substitution $0 \mapsto B_1$, $1 \mapsto B_2$.

We also show that such blocks B_1 and B_2 cannot be found for *all* quadratic irrationals α.

However, we show that it is true that for every quadratic irrational α, f_α is invariant under a substitution of the following kind. There always exist blocks s,t,C_1,C_2 of 0's and 1's (where C_1 is longer than s or C_2 is longer than t) such that f_α may be written as a sequence of s's and t's, C_1 and C_2 may be written as blocks of s's and t's, and f_α is invariant under the substitution $s \mapsto C_1$, $t \mapsto C_2$.

Finally, we show that if f_α is invariant under a substituion in the above sense, then α is a quadratic irrational. Thus the 'substitution property' of f_α characterizes quadratics among the irrationals.

2 Results

We will make use of the sequence $g_\alpha = (g_\alpha(n) : n \geq 1)$, the characteristic function of the sequence $([n\alpha] : n \geq 1)$, where $g_\alpha(n) = 1$ if $n = [k\alpha]$ for some k, and $g_\alpha(n) = 0$ otherwise. We will also use the fact (from the defnition of f_α) that if j is any integer with $0 \leq j < \alpha$, then $f_\alpha = f_{\alpha-j}$.

It will be convenient to use the notation $1/b = [0,b]$, $1/(a+1/b) = [0,a,b]$, etc., for positive real numbers a,b.

We begin with a fact which is mentioned by Fraenkel, Mushkin, and Tassa [3]:

Lemma. *For any irrational $\alpha > 1$, $g_\alpha = f_{1/\alpha}$.*

Proof. It is straightforward to show from the definitions of g_α and $f_{1/\alpha}$ that $g_\alpha(n) = 1 \Rightarrow f_{1/\alpha}(n) = 1$ and $g_\alpha(n) = 0 \Rightarrow f_{1/\alpha}(n) = 0$. □

Definition 1. Let $k \geq 1$ be fixed, and let w be any block of 0's and 1's or any sequence of 0's or 1's. Then $h_k(w)$ is obtained from w by applying the substitution $0 \mapsto 0^{k-1}1$, $1 \mapsto 0^{k-1}10$, where 0^{k-1} is a block of $k-1$ consecutive 0's. That is, $h_k(w)$ is obtained from w by replacing each 0 in w by $0^{k-1}1$, and each 1 by $0^{k-1}10$. If $k = 1$ the substitution is $0 \mapsto 1$, $1 \mapsto 10$.

Lemma. *Let $k \geq 1$ and α be given, where α is irrational and $0 < \alpha < 1$. Then $h_k(f_\alpha) = g_{k+\alpha} = f_{1/(k+\alpha)}$.*

Proof. By definition,

$$f_\alpha = f_\alpha(1)f_\alpha(2)\cdots f_\alpha(j)\cdots,$$

where

$$f_\alpha(j) = [(j+1)\alpha] - [j\alpha], \quad j \geq 1,$$

and

$$h_k(\alpha) = D_1 D_2 \cdots D_q D_{q+1}\cdots,$$

where

$$D_j = h_k(f_\alpha(j)), \quad j \geq 1.$$

Note that each block D_j contains exactly one "1", which is in the kth position, and has length either k or $k+1$.

Consider $h_k(f_\alpha)$ now as a sequence of 0's and 1's, and let n be the position in this sequence of the $(q+1)^{\text{st}}$ "1", that is, the 1 in the block D_{q+1}. Then

$$n = L(D_1 D_2 \cdots D_q) + k,$$

where $L(D_1D_2\cdots D_q)$ denotes the *length* of $D_1D_2\cdots D_q$.

Since the block D_j has length k if $f_\alpha(j)=0$ and has length $k+1$ if $f_\alpha(j)=1$, it follows that

$$L(D_1D_2\cdots D_q)=qk+f_\alpha(1)+\cdots+f_\alpha(q).$$

Since $f_\alpha(j)=[(j+1)\alpha]-[j\alpha]$, and $[\alpha]=0$, the sum telescopes to

$$L(D_1D_2\cdots D_q)=qk+[(q+1)\alpha].$$

Thus n, the position of the $(q+1)^{\text{st}}$ "1" in the sequence $h_k(f_\alpha)$ satisfies

$$n=qk+[(q+1)\alpha]+k=[(q+1)(k+\alpha)].$$

Thus $[h_k(f_\alpha)(n)=1]\Leftrightarrow[n=[(q+1)(k+\alpha)]$ for some $q\geq 0]\Leftrightarrow[g_{k+\alpha}(n)=1]$. That is, $h_k(f_\alpha)=g_{k+\alpha}$.

Using Lemma 2, this gives

$$h_k(f_\alpha)=g_{k+\alpha}=f_{1/(k+\alpha)}.$$ □

Theorem 1. *Let* $\alpha=[0,\overline{a_1,\ldots,a_m}]$. *Then* f_α *is invariant under the substitution*

$$\begin{aligned}0&\mapsto B_1=h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_m}(0)\\1&\mapsto B_2=h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_m}(1),\end{aligned}$$

where $\circ$ *denotes composition.*

Proof. By Lemma 2, we have $h_{a_m}(f_\alpha)=f_{1/(a_m+\alpha)}=f_{[0,a_m+\alpha]}$, $h_{a_{m-1}}\circ h_{a_m}(f_\alpha)=f_{1/(a_{m-1}+[0,a_m+\alpha])}=f_{[0,a_{m-1},a_m+\alpha]},\ldots,h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_m}(f_\alpha)=f_\beta$, where $\beta=[0,a_1,a_2,\ldots,a_m+\alpha]=\alpha$. □

Remark. The blocks $B_1=h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_m}(0)$ and $B_2=h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_m}(1)$ can be described as follows. Let $S_0=0$, $S_1=0^{a_1-1}1$, and for $2\leq j\leq m$, let $S_j=(S_{j-1})^{a_j}S_{j-2}$. Then $B_1=S_m$, and $B_2=S_mS_{m-1}$. This can be seen by induction on m.

Corollary. *For any block* w *of 0's and 1's, let* $H(w)$ *be obtained from* w *by replacing each 0 by* B_1*, and each 1 by* B_2*. (That is,* $H=h_{a_1}\circ h_{a_2}\circ\cdots\circ h_{a_m}$*.) Then* f_α *can be generated by starting with* $w=f_\alpha(1)$ *and repeatedly replacing* w *by* $H(w)$*.*

Proof. Let $E_1=f_\alpha(1)$ and for $k\geq 1$ let $E_{k+1}=H(E_k)$. Then by the Theorem and induction, each E_k is an initial segment of f_α. □

Theorem 2. *Let* $\beta>0$ *be any quadratic irrational. Since* $f_\beta=f_{\beta-[\beta]}$*, assume without loss of generality that* $0<\beta<1$*, so that (for suitable* a_i,b_j*)* $\beta=[0,b_1,\ldots,b_q,\overline{a_1,\ldots,a_m}]$*. Let*

$$\begin{aligned}s&=h_{b_1}\circ h_{b_2}\circ\cdots\circ h_{b_q}(0)\\t&=h_{b_1}\circ h_{b_2}\circ\cdots\circ h_{b_q}(1),\end{aligned}$$

and

$$C_1 = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q} \circ h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(0)$$
$$C_2 = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q} \circ h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(1).$$

Then f_β, C_1, C_2, can be written as sequences of s's and t's and f_β is invariant under the substitution $s \mapsto C_1, t \mapsto C_2$.

Proof. Let $\alpha = [0, \overline{a_1, \ldots, a_m}]$, so that $\beta = [0, b_1, b_2, \ldots, b_q + \alpha]$. Let $H_1 = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q}$, $H_2 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}$, $B_1 = H_2(0)$, $B_2 = H_2(1)$. Then we have $s = H_1(0)$, $t = H_1(1)$, and $C_1 = H_1(B_1)$, $C_2 = H_1(B_2)$, so that C_1, C_2 can be written as blocks of s's and t's.

By the proof of Theorem 1, $H_1(f_\alpha) = f_\beta$, so that f_β can be written as a sequence of s's and t's, and $H_2(f_\alpha) = f_\alpha$. Note that $H_1(f_\alpha) = f_\beta$ is obtained from f_α by applying the substitution $0 \mapsto s, 1 \mapsto t$, and $H_2(f_\alpha) = f_\alpha$ is obtained by applying the substituion $0 \mapsto B_1, 1 \mapsto B_2$.

Therefore we can transfrom f_β into f_β by successively applying the substitutions $[s \mapsto 0, t \mapsto 1]$, $[0 \mapsto B_1, 1 \mapsto B_2]$, and $[B_1 \mapsto C_1, B_2 \mapsto C_2]$, which transform f_β into f_α, f_α into f_α, and f_α into f_β respectively. □

Theorem 3. *Let $\beta = [0, 5, 1, 1, 1, \ldots]$. Then there do not exist non-trivial blocks B_1, B_2 of 0's and 1's such that f_β is invariant under the substitution $0 \mapsto B_1, 1 \mapsto B_2$.*

Proof. Let $\alpha = [0, 1, 1, \ldots] = (\sqrt{5} - 1)/2$. Then $f_\alpha(1) = 1$, and by Theorem 1, f_α is invariant under $0 \mapsto 1, 1 \mapsto 10$, so $f_\alpha = 10110\cdots$. By the proof of Theorem 1, $h_5(f_\alpha) = f_\beta$, so $f_\beta = tstts\cdots$, where $s = 00001$, $t = 000010$. Thus

$$\begin{aligned} f_\beta &= 000010\;00001\;000010\;000010\;00001\;000010\;00001\;000010\cdots \\ &= \quad t \qquad s \qquad t \qquad t \qquad s \qquad t \qquad s \qquad t\cdots. \end{aligned}$$

It was shown by Karhumaki [4] that the sequence $f_\alpha = 10110\cdots$ does not contain any 4th power. That is, f_α contains no non-empty block of the form $DDDD$. (Se also [2] and [6].) Thus also the sequence $f_\beta = tsttstst\cdots$, *when regarded as a sequence of s's and t's*, contains non non-empty block of the form $DDDD$, *where D is a block of s's and t's*.

Now suppose that f_β is invariant under $0 \mapsto B_1$, $1 \mapsto B_2$, and for any block or sequence w of 0's and 1's, let $H(w)$ be the result of applying this substitution to w. Note that $f_\beta = H(f_\beta) = B_1B_1B_1B_1B_2\cdots$, so that B_1 is some initial segment of f_β.

Our goal is to show that B_1 can be written as a block of s's and t's, which form an initial segment of f_β, *when f_β is regarded as a sequence of s's and t's*. Since $f_\beta = B_1B_1B_1\cdots$, this will contradict Karhumaki's result, and the proof will be finished.

To this end, first note that the block B_1 must contain at least one "1", since otherwise, by the Corollary to Theorem 1, f_β would be identically 0. Note also that f_β consists of blocks of either 4 or 5 consecutive 0's separated by single 1's. This implies that the block B_1 ends either with 1 or with 10, since otherwise $f_\beta = H(f_\beta) = B_1B_1\cdots$ would contain a block of more than five consecutive 0's. (For example if $B_1 = C100$, then since $C = 00001D$ (which is true since B_1 is an initial segment of $f_\beta = 000010\cdots$), we would have

$$f_\beta = H(f_\beta) = B_1B_1\cdots = C100C100\cdots = C10000001D100\cdots,$$

with too many consecutive 0's.)

Suppose now that B_1 *fails* to be an initial segment of s's and t's in f_β (when f_β is regarded as a sequence of s's and t's).

If B_1 ends in 0, then $B1 = Cs0$, where Cs is an initial segment of s's and t's in f_β (when f_β is regarded as a sequence of s's and t's). Note that $C = 00001D$. Then since Cs is an initial segment of f_β, we have $f_\beta = Cs\underline{0000}1\cdots$, but also we have

$$f_\beta = H(f_\beta) = B_1B_1\cdots = Cs0Cs0\cdots = Cs\underline{0\,00001}Ds0\cdots,$$

a contradiction.

If B_1 ends in 1, then $B_1 = C00001$, where Ct is an initial segment of s's and t's in f_β (when f_β is regarded as a sequence of s's and t's). Note again that $C = 00001D$. Then since Ct is an initial segment of f_β, we have $f_\beta = Ct\cdots = C00001\underline{0\,00001}\cdots$, but also we have

$$f_\beta = H(f_\beta) = B_1B_1\cdots = C00001C00001\cdots = C00001\underline{0000}1D00001\cdots,$$

a contradiction.

Thus we have shown that if f_β is invariant under a substitution $0 \mapsto B_1, 1 \mapsto B_2$, then B_1 can be written as a block of s's and t's, which form an initial segment of f_β, *when f_β is regarded as a sequence of s's and t's*. This gives the desired contradiction to Karhumaki's result, and completes the proof. □

Theorem 4. *Let α be a positive irrational real number, and let f_α be written as a sequence on s,t, where s,t are blocks of 0's and 1's. Let C_1, C_2 be blocks of s's and t's such that f_α is invariant under the non-trivial substitution $s \mapsto C_1, t \mapsto C_2$. Then α is a quadratic irrational.*

Proof. First consider the case where $0 < \alpha < 1$ and $s = 0$, $t = 1$. Suppose that C_1 contains a 0's and b 1's, and that C_2 contains c 0's and d 1's.

For any block w of 0's and 1's, let $H(w)$ denote the word obtained from w by replacing each 0 by C_1 and each 1 by C_2. Let $E_1 = H(f_\alpha(1))$ and for $p \geq 1$ let $E_{p+1} = H(E_p)$. Let e_p denote the number of 1's which occur in the block E_p. Since the number of 1's which occur in the block $f_\alpha(1)f_\alpha(2)\cdots f_\alpha(n)$ is $f_\alpha(1) + f_\alpha(2) + \cdots f_\alpha(n)$, which equals $[(n+1)\alpha]$, we have $e_p = [(L(E_p)+1)\alpha]$. Also, $e_{p+1} = e_p d + (L(E_p) - e_p)b$, and $L(E_{p+1}) = e_p(c+d) + (L(E_p) - e_p)(a+b)$, so that

$$\frac{e_{p+1}}{L(E_{p+1})} = \frac{\frac{e_p d}{L(E_p)} + (1 - \frac{e_p}{L(E_p)})b}{\frac{e_p(c+d)}{L(E_p)} + (1 - \frac{e_p}{L(E_p)})(a+b)}.$$

Taking the limit as $p \to \infty$, we obtain

$$\alpha = \frac{\alpha + (1-\alpha)b}{\alpha(c+d) + (1-\alpha)(a+b)},$$

so that α is a quadratic irrational.

The general case can be handled similarly. We omit the details. □

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