Monochromatic Solutions to Equations with Unit Fractions

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Our main result is that if $G(x_1, \ldots, x_n) = 0$ is a system of homogeneous equations such that for every partition of the positive integers into finitely many classes there are distinct $y_1, \ldots, y_n$ in one class such that $G(x_1, \ldots, x_n) = 0$, then, for every partition of the positive integers into finitely many classes there are distinct $z_1, \ldots, z_n$ in one class such that

$$G\left(\frac{1}{z_1}, \ldots, \frac{1}{z_n}\right) = 0.$$ 

In particular, we show that if the positive integers are split into $r$ classes, then for every $n \geq 2$ there are distinct positive integers $x_0, x_1, \ldots, x_n$ in one class such that

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$ 

We also show that if $\left[1, n^6 - (n^2 - n)^2\right]$ is partitioned into two classes, then some class contains $x_0, x_1, \ldots, x_n$ such that

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$ 

(Here $x_0, x_1, \ldots, x_n$ are not necessarily distinct.)

1 Introduction

In their monograph [1], Erdős and Graham list a large number of questions concerned with equations with unit fractions. In fact, a whole chapter is devoted to this topic. One of their questions, still open, is the following.

In the positive integers, let

$$H_m = \left\{ \{x_1, \ldots, x_m\} : \sum_{k=1}^{m} 1/x_k = 1, 0 < x_1 < \cdots < x_m \right\},$$

and let $H$ denote the union of all the $H_m, m \geq 1$. Now arbitrarily split the positive integers into $r$ classes. Is it true that some element of $H$ is contained entirely in one class?

Received 29 May 1990

The first author was partially supported by NSERC.
In this note we show (Corollary 2.3 below) that if one does not require all the \(x_k\)'s to be distinct, but only many of the \(x_k\)'s to be distinct, then the answer to the corresponding question is yes. More precisely, we show that if the positive integers are split into \(r\) classes, then for every \(n\) there exist \(m \geq n\) and \(x_1, \ldots, x_m\) (not necessarily distinct) in one class such that \(\{x_1, \ldots, x_m\}\) \(\geq n\) and \(\Sigma_{k=1}^{m} 1/x_k = 1\).

We actually show (Corollary 2.2 below) something stronger, namely that if the positive integers are split into \(r\) classes, then for every \(n \geq 2\) there are distinct positive integers \(x_0, x_1, \ldots, x_n\) in one class such that
\[
\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.
\]
(The preceding result then follows by taking \(x_0\) copies of each of \(x_1, \ldots, x_n\).)

Our main result (Theorem 2.1) is that if \(G(x_1, \ldots, x_n) = 0\) is a system of homogeneous equations such that for every partition of the positive integers into finitely many classes there are distinct \(y_1, \ldots, y_n\) in one class such that \(G(x_1, \ldots, x_n) = 0\), then, for every partition of the positive integers into finitely many classes there are distinct \(z_1, \ldots, z_n\) in one class such that
\[
G\left(\frac{1}{z_1}, \ldots, \frac{1}{z_n}\right) = 0.
\]

In particular, we show that if the positive integers are split into \(r\) classes, then for every \(n \geq 2\) there are distinct positive integers \(x_0, x_1, \ldots, x_n\) in one class such that
\[
\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.
\]

We also prove (Theorem 2.3) the following quantitative result. Let \(f(n)\) be the smallest \(N\) such that if \([1, N]\) is partitioned into two classes, then some class contains \(x_0, x_1, \ldots, x_n\) such that \(1/x_0 = 1/x_1 + \cdots + 1/x_n\). (Here, \(x_0, x_1, \ldots, x_n\) are not necessarily distinct.) Then
\[
f(n) \leq n^6 - (n^2 - n)^2.
\]

2 Results

From now on we shall use the terminology of colourings rather than partitions. That is, instead of “partition into \(r\)” classes we say “\(r\)-colouring,” and instead of “there are distinct \(y_1, \ldots, y_n\) in one class such that \(G(y_1, \ldots, y_n) = 0\)” we say “there is a monochromatic solution of \(G(y_1, \ldots, y_n) = 0\) in distinct \(y_1, \ldots, y_n\)”.

**Theorem 2.1.** Let \(G(x_1, \ldots, x_n) = 0\) be a system of homogeneous equations such that for every finite colouring of the positive integers there is a monochromatic solution of \(G(y_1, \ldots, y_n) = 0\) in distinct \(y_1, \ldots, y_n\). Then for every finite colouring of the positive integers there is a monochromatic solution of \(G(1/z_1, \ldots, 1/z_n) = 0\) in distinct \(z_1, \ldots, z_n\).

**Proof.** Let \(r\) be given, and consider a system \(G(x_1, \ldots, x_n) = 0\) of homogeneous equations such that for every \(r\)-colouring of the positive integers there is a monochromatic solution of \(G(y_1, \ldots, y_n) = 0\) in distinct \(y_1, \ldots, y_n\). By a standard compactness argument, there exists a positive integer \(T\) such that if
[1, T] is r-coloured, there is a monochromatic solution to $G(y_1, \ldots, y_n) = 0$ in distinct $y_1, \ldots, y_n$

Let $S$ denote the least common multiple of 1, 2, $\ldots$, $T$. We show that for every $r$-colouring of [1, $S$] there is a monochromatic solution of $G(1/z_1, \ldots, 1/z_n) = 0$ in distinct $z_1, \ldots, z_n$

To do this, let

$$c : [1, S] \to [1, r]$$

be an arbitrary $r$-colouring of [1, $S$].

Define an $r$-colouring $\overline{c}$ of [1, $T$] by setting

$$\overline{c}(x) = c(S/x), 1 \leq x \leq T.$$ 

By the definition of $T$, there is a solution of $G(y_1, \ldots, y_n) = 0$ in distinct $y_1, \ldots, y_n$ such that

$$\overline{c}(y_1) = \overline{c}(y_2) = \cdots = \overline{c}(y_n).$$

By the definition of $\overline{c}$, this means that

$$c(S/y_1) = c(S/y_2) = \cdots = c(S/y_n).$$

Setting $z_i = S/y_i, 1 \leq i \leq n$, we have that $z_1, \ldots, z_n$ are distinct, are monochromatic relative to the colouring $c$ of [1, $S$], and that

$$G \left( \frac{1}{z_1}, \ldots, \frac{1}{z_n} \right) = 0.$$  \hspace{1cm} \square

Omitting all references to distinctness, one gets the following.

**Theorem 2.2.** Let $G(x_1, \ldots, x_n) = 0$ be a system of homogeneous equations such that for every finite colouring of the positive integers there is a monochromatic solution of $G(x_1, \ldots, x_n) = 0$. Then, for every finite colouring of the positive integers there is monochromatic solution of $G(1/z_1, \ldots, 1/z_n) = 0$.

**Corollary 2.1.** Let $a_1, \ldots, a_m, b_1, \ldots, b_n$ be positive integers such that

1. some non-empty subset of the $a_i$'s has the same sum as some non-empty subset of the $b_j$'s and
2. there exist distinct integers $u_1, \ldots, u_m, v_1, \ldots, v_n$ such that $a_1u_1 + \cdots + a_mu_m = b_1v_1 + \cdots + b_nv_n$.

Then, given any $r$-colouring of the positive integers, there is a monochromatic solution of

$$\frac{a_1}{x_1} + \cdots + \frac{a_m}{x_m} = \frac{b_1}{y_1} + \cdots + \frac{b_n}{y_n}$$

in distinct $x_1, \ldots, x_m, y_1, \ldots, y_n$.

**Proof.** Let $a_1, \ldots, a_m, b_1, \ldots, b_n$ satisfy conditions 1. and 2. According to Rado’s theorem [3] (also see [2, p. 59]), the equation

$$a_1x_1 + \cdots + a_mx_m = b_1y_1 + \cdots + b_ny_n$$

will always have a monochromatic solution $x_1, \ldots, x_m, y_1, \ldots, y_n$, for every $r$-colouring of the positive integers, because of condition 1. The additional condition 2. is enough (see [2, p. 62 Corollary 8.2]) to
ensure that the equation
\[ a_1 x_1 + \cdots + a_m x_m = b_1 y_1 + \cdots + b_n y_n \]
will always have a monochromatic solution \( x_1, \ldots, x_m, y_1, \ldots, y_n \), in distinct \( x_1, \ldots, x_m, y_1, \ldots, y_n \). Theorem 2.1 now applies.

Corollary 2.2. Let an arbitrary \( r \)-colouring of the positive integers be given. Let \( n, a \) be positive integers, with \( n \geq 2 \), and \( 1 \leq a \leq n \). Then the equation
\[
\frac{a}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}
\]
has a monochromatic solution in distinct \( x_0, x_1, \ldots, x_n \).

Proof. This follows immediately from Corollary 2.1.

Corollary 2.3. Let an arbitrary colouring of the positive integers be given. Then for every \( n \) there exist \( m \geq n \) and monochromatic \( x_1, \ldots, x_m \) (not necessarily distinct) such that \( \{x_1, \ldots, x_m\} \geq n \) and \( \sum_{k=1}^{m} 1/x_k = 1 \).

Proof. Apply Corollary 2.2 (with \( a = 1 \)) and take \( x_0 \) copies of each of \( x_1, \ldots, x_m \).

Theorem 2.3. For each \( n \geq 2 \), let \( f(n) \) be the smallest \( N \) such that if \( [1, N] \) is partitioned into two classes, then some class contains \( x_0, x_1, \ldots, x_n \) such that
\[
\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.
\]
(Here, \( x_0, x_1, \ldots, x_n \) are not necessarily distinct.) Then
\[
f(n) \leq n^6 - (n^2 - n)^2.
\]

Proof. The proof is by contradiction. Fix \( n \geq 2 \), let \( N = n^6 - (n^2 - n)^2 \), and suppose throughout the proof that \( c : [1, N] \mapsto \{1, 2\} \) is some fixed 2-colouring of \( [1, N] \) for which there does not exist any monochromatic solution of
\[
\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.
\]

Lemma 2.1. (a) If \( nx \leq N \) then \( c(nx) \neq c(x) \).

(b) If \( n^2 x \leq N \) then \( c(n^2 x) = c(x) \).

Proof. Part (a) follows from \( 1/x = 1/(nx) + \cdots + 1/(nx) \). Part (b) follows from part (a).

Lemma 2.2. If \( n^2 (n^2 + n - 1) x \leq N \), then \( c((n^2 + n - 1) x) \neq c(x) \).

Proof. This follows from
\[
\frac{1}{n^2 x} = \frac{1}{n^2 + n - 1} x + \frac{1}{n^2(n^2 + n - 1) x}
\]
and Lemma 2.1.
Lemma 2.3. If $n^2(n^2 - n + 1)x \leq N$, then $c((n^2 - n + 1)x) \neq c(x)$.

Proof. This follows from
\[
\frac{1}{(n^2-n+1)x} = \frac{1}{n^2x} + (n-1)\frac{1}{n^2(n^2-n+1)x}
\]
and Lemma 2.1.

\[
\tag*{□}
\]

Lemma 2.4. If $n^2(n^2 + n - 1)x \leq N$, then $c((n+1)x) = c(x)$.

Proof. This follows from
\[
\frac{1}{n(n+1)x} = \frac{1}{(n^2+n-1)(n+1)x} + (n-1)\frac{1}{(n^2+n-1)nx},
\]
and Lemmas 2.1 and 2.2.

\[
\tag*{□}
\]

Lemma 2.5. If $n^2(n^2 + n - 1)(n^2 - n + 1)x \leq N$, then $c(2x) = c(x)$.

Proof. This follows from
\[
\frac{1}{(n^2-n+1)2x} = \frac{1}{(n^2+n-1)2x} + (n-1)\frac{1}{(n^2+n-1)(n^2-n+1)x}
\]
and Lemmas 2.2 and 2.3.

\[
\tag*{□}
\]

Finally, Theorem 2.3 is proved by observing that
\[
\frac{1}{2} = \frac{1}{(n+1)\cdot 1} + (n-1)\frac{1}{2(n+1)\cdot 1},
\]
and by Lemmas 2.4 and 2.5, $c(2 \cdot 1) = c((n+1)\cdot 1) = c(2(n+1)\cdot 1) = c(1)$, a contradiction.

\[
\tag*{□}
\]

Remark. The authors have learned that Hanno Lefmann (Bielefeld) has independently obtained results which include our Theorem 2.2.

References

