Monochromatic Solutions to Equations with Unit Fractions

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Our main result is that if $G(x_1,\dots,x_n)=0$ is a system of homogeneous equations such that for every partition of the positive integers into finitely many classes there are distinct $y_1,\dots,y_n$ in one class such that $G(x_1,\dots,x_n)=0$, then, for every partition of the positive integers into finitely many classes there are distinct $z_1,\dots,z_n$ in one class such that

$$G\left(\frac{1}{z_1},\dots,\frac{1}{z_n}\right)=0.$$

In particular, we show that if the positive integers are split into r classes, then for every $n\geq 2$ there are distinct positive integers $x_0,x_1,\dots,x_n$ in one class such that

$$\frac{1}{x_0}=\frac{1}{x_1}+\cdots+\frac{1}{x_n}.$$

We also show that if $[1,n^6-(n^2-n)^2]$ is partitioned into two classes, then some class contains $x_0,x_1,\dots,x_n$ such that

$$\frac{1}{x_0}=\frac{1}{x_1}+\cdots+\frac{1}{x_n}.$$

(Here $x_0,x_1,\dots,x_n$ are not necessarily distinct.)

1 Introduction

In their monograph [1], Erdős and Graham list a large number of questions concerned with equations with unit fractions. In fact, a whole chapter is devoted to this topic. One of their questions, still open, is the following.

In the positive integers, let

$$H_m=\left\{\{x_1,\dots,x_m\}:\sum_{k=1}^{m}1/x_k=1,0<x_1<\cdots<x_m\right\},$$

and let H denote the union of all the $H_m, m\geq 1$. Now arbitrarily split the positive integers into r classes. Is it true that some element of H is contained entirely in one class?

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In this note we show (Corollary 2.3 below) that if one does not require *all* the x_k's to be distinct, but only *many* of the x_k's to be distinct, then the answer to the corresponding question is yes. More precisely, we show that if the positive integers are split into r classes, then for every n there exist $m \geq n$ and $x_1, \ldots, x_m$ (not necessarily distinct) in one class such that $|\{x_1, \ldots, x_m\}| \geq n$ and $\sum_{k=1}^{m} 1/x_k = 1$.

We actually show (Corollary 2.2 below) something stronger, namely that if the positive integers are split into r classes, then for every $n \geq 2$ there are *distinct* positive integers $x_0, x_1, \ldots, x_n$ in one class such that

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$

(The preceding result then follows by taking x_0 copies of each of $x_1, \ldots, x_n$.)

Our main result (Theorem 2.1) is that if $G(x_1, \ldots, x_n) = 0$ is a system of homogeneous equations such that for every partition of the positive integers into finitely many classes there are distinct $y_1, \ldots, y_n$ in one class such that $G(x_1, \ldots, x_n) = 0$, then, for every partition of the positive integers into finitely many classes there are distinct $z_1, \ldots, z_n$ in one class such that

$$G\left(\frac{1}{z_1}, \ldots, \frac{1}{z_n}\right) = 0.$$

In particular, we show that if the positive integers are split into r classes, then for every $n \geq 2$ there are distinct positive integers $x_0, x_1, \ldots, x_n$ in one class such that

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$

We also prove (Theorem 2.3) the following quantitative result. Let $f(n)$ be the smallest N such that if $[1, N]$ is partitioned into *two* classes, then some class contains $x_0, x_1, \ldots, x_n$ such that $1/x_0 = 1/x_1 + \cdots + 1/x_n$. (Here, $x_0, x_1, \ldots, x_n$ are not necessarily distinct.) Then

$$f(n) \leq n^6 - (n^2 - n)^2.$$

2 Results

From now on we shall use the terminology of *colourings* rather than *partitions*. That is, instead of "partition into r" classes we say "r-colouring," and instead of "there are distinct $y_1, \ldots, y_n$ in one class such that $G(y_1, \ldots, y_n) = 0$" we say "there is a monochromatic solution of $G(y_1, \ldots, y_n) = 0$ in distinct $y_1, \ldots, y_n$".

Theorem 2.1. *Let $G(x_1, \ldots, x_n) = 0$ be a system of homogeneous equations such that for every finite colouring of the positive integers there is a monochromatic solution of $G(y_1, \ldots, y_n) = 0$ in distinct $y_1, \ldots, y_n$. Then for every finite colouring of the positive integers there is a monochromatic solution of $G(1/z_1, \ldots, 1/z_n) = 0$ in distinct $z_1, \ldots, z_n$.*

Proof. Let r be given, and consider a system $G(x_1, \ldots, x_n) = 0$ of homogeneous equations such that for every r-colouring of the positive integers there is a monochromatic solution of $G(y_1, \ldots, y_n) = 0$ in distinct $y_1, \ldots, y_n$. By a standard compactness argument, there exists a positive integer T such that if

$[1,T]$ is r-coloured, there is a monochromatic solution to $G(y_1,\ldots,y_n)=0$ in distinct $y_1,\ldots,y_n$.

Let S denote the least common multiple of $1,2,\ldots,T$. We show that for every r-colouring of $[1,S]$ there is a monochromatic solution of $G(1/z_1,\ldots,1/z_n)=0$ in distinct $z_1,\ldots,z_n$.

To do this, let

$$c:[1,S]\to[1,r]$$

be an arbitrary r-colouring of $[1,S]$.

Define an r-colouring $\bar{c}$ of $[1,T]$ by setting

$$\bar{c}(x)=c(S/x), 1\le x\le T.$$

By the definition of T, there is a solution of $G(y_1,\ldots,y_n)=0$ in distinct $y_1,\ldots,y_n$ such that

$$\bar{c}(y_1)=\bar{c}(y_2)=\cdots=\bar{c}(y_n).$$

By the definition of $\bar{c}$, this means that

$$c(S/y_1)=c(S/y_2)=\cdots=c(S/y_n).$$

Setting $z_i=S/y_i, 1\le i\le n$, we have that $z_1,\ldots,z_n$ are distinct, are monochromatic relative to the colouring c of $[1,S]$, and that

$$G\left(\frac{1}{z_1},\ldots,\frac{1}{z_n}\right)=0. \qquad \square$$

Omitting all references to distinctness, one gets the following.

Theorem 2.2. *Let $G(x_1,\ldots,x_n)=0$ be a system of homogeneous equations such that for every finite colouring of the positive integers there is a monochromatic solution of $G(x_1,\ldots,x_n)=0$. Then, for every finite colouring of the positive integers there is monochromatic solution of $G(1/z_1,\ldots,1/z_n)=0$.*

Corollary 2.1. *Let $a_1,\ldots,a_m,b_1,\ldots,b_n$ be positive integers such that*

1. *some non-empty subset of the a_i's has the same sum as some non-empty subset of the b_j's and*
2. *there exist distinct integers $u_1,\ldots,u_m,v_1,\ldots,v_n$ such that $a_1u_1+\cdots+a_mu_m=b_1v_1+\cdots+b_nv_n$.*

Then, given any r-colouring of the positive integers, there is a monochromatic solution of

$$\frac{a_1}{x_1}+\cdots+\frac{a_m}{x_m}=\frac{b_1}{x_1}+\cdots+\frac{b_n}{y_n}.$$

in distinct $x_1,\ldots,x_m,y_1,\ldots,y_n$.

Proof. Let $a_1,\ldots,a_m,b_1,\ldots,b_n$ satisfy conditions 1. and 2. According to Rado's theorem [3] (also see [2, p. 59]), the equation

$$a_1x_1+\cdots+a_mx_m=b_1y_1+\cdots+b_ny_n$$

will always have a monochromatic solution $x_1,\ldots,x_m,y_1,\ldots,y_n$, for every r-colouring of the positive integers, because of condition 1. The additional condition 2. is enough (see [2, p. 62 Corollary $8\frac{1}{2}$]) to ensure that the equation

$$a_1x_1 + \cdots + a_mx_m = b_1y_1 + \cdots + b_ny_n$$

will always have a monochromatic solution $x_1, \ldots, x_m$, $y_1, \ldots, y_n$, in *distinct* $x_1, \ldots, x_m, y_1, \ldots, y_n$. Theorem 2.1 now applies. □

Corollary 2.2. *Let an arbitrary r-colouring of the positive integers be given. Let n, a be positive integers, with $n \geq 2$, and $1 \leq a \leq n$. Then the equation*

$$\frac{a}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

has a monochromatic solution in distinct $x_0, x_1, \ldots, x_n$.

Proof. This follows immediately from Corollary 2.1. □

Corollary 2.3. *Let an arbitrary colouring of the positive integers be given. Then for every n there exist $m \geq n$ and monochromatic $x_1, \ldots, x_m$ (not necessarily distinct) such that $|\{x_1, \ldots, x_m\}| \geq n$ and $\sum_{k=1}^{m} 1/x_k = 1$.*

Proof. Apply Corollary 2.2 (with $a = 1$) and take x_0 copies of each of $x_1, \ldots, x_m$. □

Theorem 2.3. *For each $n \geq 2$, let $f(n)$ be the smallest N such that if $[1, N]$ is partitioned into two classes, then some class contains $x_0, x_1, \ldots, x_n$ such that*

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$

(Here, $x_0, x_1, \ldots, x_n$ are not necessarily distinct.) Then

$$f(n) \leq n^6 - (n^2 - n)^2.$$

Proof. The proof is by contradiction. Fix $n \geq 2$, let $N = n^6 - (n^2 - n)^2$, and suppose throughout the proof that $c : [1, N] \mapsto \{1, 2\}$ is some fixed 2-colouring of $[1, N]$ for which there does *not* exist any monochromatic solution of

$$\frac{1}{x_0} = \frac{1}{x_1} + \cdots + \frac{1}{x_n}.$$

Lemma 2.1. *(a) If $nx \leq N$ then $c(nx) \neq c(x)$.*
(b) If $n^2x \leq N$ then $c(n^2x) = c(x)$.

Proof. Part (a) follows from $1/x = 1/(nx) + \cdots + 1/(nx)$. Part (b) follows from part (a). □

Lemma 2.2. *If $n^2(n^2 + n - 1)x \leq N$, then $c((n^2 + n - 1)x) \neq c(x)$.*

Proof. This follows from

$$\frac{1}{n^2x} = \frac{1}{n^2 + n - 1}x + (n-1)\frac{1}{n^2(n^2 + n - 1)x}$$

and Lemma 2.1. □

Lemma 2.3. *If $n^2(n^2-n+1)x \le N$, then $c((n^2-n+1)x) \ne c(x)$.*

Proof. This follows from

$$\frac{1}{(n^2-n+1)x} = \frac{1}{n^2x} + (n-1)\frac{1}{n^2(n^2-n+1)x}$$

and Lemma 2.1. □

Lemma 2.4. *If $n^2(n^2+n-1)x \le N$, then $c((n+1)x) = c(x)$.*

Proof. This follows from

$$\frac{1}{n(n+1)x} = \frac{1}{(n^2+n-1)(n+1)x} + (n-1)\frac{1}{(n^2+n-1)nx},$$

and Lemmas 2.1 and 2.2. □

Lemma 2.5. *If $n^2(n^2+n-1)(n^2-n+1)x \le N$, then $c(2x) = c(x)$.*

Proof. This follows from

$$\frac{1}{(n^2-n+1)2x} = \frac{1}{(n^2+n-1)2x} + (n-1)\frac{1}{(n^2+n-1)(n^2-n+1)x}$$

and Lemmas 2.2 and 2.3. □

Finally, Theorem 2.3 is proved by observing that

$$\frac{1}{2\cdot 1} = \frac{1}{(n+1)\cdot 1} + (n-1)\frac{1}{2(n+1)\cdot 1},$$

and by Lemmas 2.4 and 2.5, $c(2\cdot 1) = c((n+1)\cdot 1) = c(2(n+1)\cdot 1) = c(1)$, a contradiction. □

Remark. The authors have learned that Hanno Lefmann (Bielefeld) has independently obtained results which include our Theorem 2.2.

References

[1] P. Erdős and R.L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], vol. 28, Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[2] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, *Ramsey theory*, Wiley-Interscience Series in Discrete Mathematics. A Wiley-Interscience Publication., John Wiley & Sons, Inc., New York, 1980.

[3] R. Rado, *Studien zur kombinatorik*, Math Z. **36** (1933), 424–480.