Cancellation in Semigroups in Which $x^2 = x^3$

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1 Introduction

Let $B(k,m,n)$ denote the semigroup generated by k elements and satisfying the identity $x^m = x^n$, where $0 \leq m < n$. That is, $B(k,m,n)$ is the free semigroup on k generators in the variety of semigroups defined by the law $x^m = x^n$ (we are following the notation of Lallement [7]).

Green and Rees [6] showed that for each $n \geq 1$, the semigroups $B(k,1,n)$ are finite for all $k \geq 1$ if and only if the groups $B(k,0,n-1)$ are finite for all $k \geq 1$. Thus in particular all semigroups in which $x = x^2$ are locally finite, and so are all semigroups in which $x = x^3$. The word problem for semigroups in which $x = x^3$ was solved by Gerhard [5].

The existence of an infinite sequence on 3 symbols in which there are no two consecutive indentical blocks shows that $B(3,2,3)$ is infinite, since the left factors of such a sequence will all be distinct modulo the law $x^2 = x^3$. This was first observed by Morse and Hedlund [9], who constructed such a sequence. An earlier construction of such a sequence was given by Thue [11], and other constructions appear in Dean [3], Dejean [4], and Leech [8].

It is much more difficult to show that $B(2,2,3)$ is infinite. This was done by Brzozowski, Culik II, and Gabrielian [2], and is also described in Lallement [7]. It follows that $B(k,m,n)$ is infinite for all $k \geq 2$, $n > m \geq 2$, since $B(k,2,3)$ is a quotient of $B(k,m,n)$.

It was shown in [1] that $S = B(k,2,3)$ is the disjoint union of locally finite subsemigroups. Specifically, for each idempotent $e = e^2$ in S, let $S_e = \{x \in S : x^2 = e\}$. Then S is the union of the locally finite subsemigroups S_e.

It has been asserted [10] that each S_e is in fact a *finite* subsemigroup of S. As far as the author knows, no proof of this assertion has been published. One possible approach of such a proof would be to show that if x is any element of S and g is any generator of S, no too much 'cancellation' can occur in the product gx.

To make this precise, for each x in S let $|x|$ denote the *length* of x, that is, the smallest integer p such that x can be written as the product of p (not necessarily distinct) generators of S.

Then if there exists a constant $c > 0$ such that $|gx| \geq c|x|$ for every generator g of S and every element x of S, it would follow that S_e is finite, for if $|e| = t$ and $x \in S_e$, then $e = ex$, so $t = |ex| \geq c^t|x|$, which bounds the length of x.

What could be the largest possible numerical value of such a constant c? This is the subject of the present note.

2 An Upper Bound for the Constant c

Since the value of c depends on k, the number of generators of S, we make the following definition.

Definition. Let $g_1, g_2, \ldots, g_k, \ldots,$ be a sequence so that for each $k \geq 1$, $g_1, g_2, \ldots, g_k$ is a set of generators for the semigroup $B_k = B(k,2,3)$, and let $B_\omega = B_1 \cup B_2 \cup \cdots$. For each $k \geq 1$, let c_k be the largest real number such that for all $x \in B_k$ and all i, $1 \leq i \leq k$, $|g_i x| \geq c_k |x|$. Similarly, let c_ω be the largest real number such that for all $x \in B_\omega$ and all $i \geq 1$, $|g_i x| \geq c_\omega |x|$.

Note that if C_k is the set of all real numbers c such that $|g_i x| \geq c|x|$ for $1 \leq i \leq k$ and $x \in B_k$, then $c_k = \sup C_k = \max C_k$.

Since $2 = |g_1(g_1)^2| \geq c_1|(g_1)^2| = 2c_1$, we have $1 = c_1$, and since $B_\omega \supseteq B_{k+1} \supseteq B_k$, we have $c_k \geq c_{k+1} \geq c_\omega$, so that

$$1 = c_1 \geq c_2 \geq \cdots \geq c_k \geq c_{k+1} \geq \cdots \geq c_\omega \geq 0.$$

It is easy to see that $2/3 \geq c_\omega$. For let $A = g_2 g_3 \cdots g_p$ nd $x = Ag_1Ag_1A$. Then $|x| = 3p-1$ and $|g_1 x| = |g_1 A g_1 A| = 2p$, so that for all $p \geq 2$,

$$\left(\frac{2}{3} + \frac{2}{9p-3}\right)|x| = |g_1 x| \geq c_\omega |x|.$$

We will improve the bound of $2/3$ to $(\sqrt{5}-1)/2 \approx .618$ by finding, for each $\varepsilon > 0$, elements x, y in B_ω so that

$$|g_1 xy^2| \leq \left(\frac{\sqrt{5}-1}{2} + \varepsilon\right)|xy^2|.$$

For this we need the following two lemmas.

Lemma 1. *Let $a,b,c \in B_\omega$, where the generator g_1 does not occur in any of a,b,c. Then $|ag_1bg_1| = |a|+|b|+2$, and (unless $a=b=c$) $|ag_1bg_1cg_1| = |a|+|b|+|c|+3$.*

Proof. For *words* X,Y in the *alphabet* $\{g_1, g_2, g_3, \ldots\}$, let us say that X is *equivalent* to Y, and write $X \approx Y$, in case X can be transformed into Y by means of a finite sequence of 'expansions' $UW^2V \to UW^3V$ and 'contractions' $UW^3V \to UW^2V$, where U,W,V are any words, possibly empty.

To prove the first equality of the lemma, suppose that $Ag_1Bg_1C = X \approx Y = Eg_1Fg_1G$, where the letter g_1 does not occur in any of the words A,B,E,F. (At this point we need not assume that g_1 does not occur in C or G; reading X from left to right, the word A is the segment of X which precedes the first occurrence of g_1, and the words B,E,F are similarly characterized.) We will show that $A \approx E$ and $B \approx F$. It suffices to consider the case where $X = Ag_1Bg_1C = UW^2V, UW^3V = Eg_1Fg_1G = Y$. By considering the several possible locations of W^2 in X (and noting that W^2 contains at least two g_1s if it contains one), one sees easily that $A \approx E$ and $B \approx F$. The fact that $|ag_1bg_1| = |a|+|b|+2$ now follows easily.

For the second equality of the lemma, we need to use also that 'right-handed' version of the preceding, namely that if $Ag_1Bg_1C \approx Eg_1Fg_1G$, where the letter g_1 does not occur in any of the words

B, C, F, G, then $B \approx F$ and $C \approx G$. Then, if the shortest product of generators which equals $ag_1bg_1cg_1$ contains at least three g_1s, it contains only three g_1s, and $|ag_1bg_1cg_1| = |a| + |b| + |c| + 3$. If the shortest such product contains only two g_1s, then it is not hard to see that $a = b = c$. □

Lemma 2. *Define elements x_n, y_n in B_ω for all $n \geq 2$ inductively as follows. Let $x_2 = g_2$, $y_2 = g_1g_2$. For $n \geq 2$, let $x_{n+1} = x_ny_ng_{n+1}$, $y_{n+1} = x_ny_n^2g_{n+1}$. Then for $n \geq 2$, $|g_1x_{n+1}y_{n+1}| = |g_1x_ny_n| + |x_ny_n^2| + 2$ and $|x_{n+1}y_{n+1}^2| = |x_ny_n| + 2|x_ny_n^2| + 3$.*

Proof. This follows from Lemma 1, with the g_{n+1} in Lemma 2 playing the role of g_1 in Lemma 1. One needs to know that $x_{n+1} \neq y_{n+1}$. But if $x_{n+1} = y_{n+1}$, then $x_ny_n = x_ny_n^2$, and this implies (by Lemma 1) that $x_{n-1}y_{n-1} = x_{n-1}y_{n-1}^2$. □

Proposition. *Let τ denote the golden mean, $\tau = (1+\sqrt{5})/2 \approx 1.618$. Then $\tau - 1 \geq c_\omega$.*

Proof. In our calculation, we will make use of the Fibonacci numbers F_n, where $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, and the fact that F_n/F_{n+1} converges to $1/\tau$.

For $n \geq 2$, let x_n, y_n be defined as in Lemma 2. Then by induction it follows that for all $n \geq 2$, $g_1x_ny_n^2 = g_1x_ny_n$. By Lemma 2 and induction it follows that for all $n \geq 2$, $|g_1x_ny_n| = F_{2n-3} + F_{2n-1}$, $|x_ny_n^2| = F_{2n-2} + F_{2n} - 2$.

Then for all $n \geq 2$,

$$c_\omega \leq \frac{|g_1x_ny_n^2|}{|x_ny_n^2|} = \frac{|g_1x_ny_n|}{|x_ny_n^2|} = \frac{F_{2n-3} + F_{2n-1}}{F_{2n-2} + F_{2n} - 2} \to 1/\tau = \tau - 1,$$

and it follows that $c_\omega \leq \tau - 1$. □

3 An Open Question

It would be interesting to know the exact values of c_2 and c_ω, and inparticular whether $c_2 > 0$, and whether $c_\omega > 0$.

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