

# On Finitely Generated Idempotent Semigroups

Tom Brown and Earl Lazerson

**Citation data:** T.C. Brown and Earl Lazerson, *On finitely generated idempotent semigroups*, Semigroup Forum **78** (2009), 183–183.

## Abstract

We give two short proofs of the well-known fact that every finitely generated idempotent semigroup is finite.

Keywords: idempotent semigroup, band, locally finite, finitely generated.

AMS Classification: 20M05

## 1 Introduction

There are at least 6 published proofs (see [1–6]) of the fact that every idempotent semigroup (i.e. semigroup in which  $x^2 = x$  for every element  $x$ ) is locally finite (every finitely generated idempotent semigroup is finite). Usually this fact is proved as a corollary to more general results. In this note we give two additional proofs which are shorter than previous proofs and which are, we believe, of independent interest. The first proof, adapted from [4], gives some standard structure theory along the way. (In particular, it shows that every idempotent semigroup is a semilattice of rectangular idempotent semigroups.) The second proof is short and self-contained, and shows directly that idempotent semigroups are locally finite.

## 2 The First Proof

**Lemma 2.1.** *Let  $S$  be an idempotent semigroup, and let  $\mathcal{T}$  denote the collection of all non-empty subsets  $T$  of  $S$  such that  $xyz = xz$  for all  $x, y, z \in T$ . Partially order  $\mathcal{T}$  by inclusion. Then every  $T \in \mathcal{T}$  is contained in a maximal element  $T'$  of  $\mathcal{T}$ . Furthermore, each maximal element of  $\mathcal{T}$  is a subsemigroup of  $S$ .*

*Proof.* Given  $T$ , Zorn's lemma shows that  $T'$  exists, and if  $x, y \in T'$ , then  $T \cup \{xy\} \in \mathcal{T}$ , so  $xy \in T'$ .  $\square$

**Lemma 2.2.** *The set of maximal elements of  $\mathcal{T}$  is a partition of  $S$ .*

*Proof.* Since  $\{x\}$  satisfies  $xyz = xz$ ,  $x$  belongs to a maximal element  $T \in \mathcal{T}$ , by Lemma 2.1, so  $\bigcup \mathcal{T} = S$ . To show that the maximal elements of  $\mathcal{T}$  are pairwise disjoint, suppose that  $T_1, T_2$  are maximal with

$e \in T_1 \cap T_2$ . Then for all  $x, y \in T_1 \cup T_2$ , as a first step

$$e(xy)e = (eye)(xy)(exe) = e(yexyex)e = e(yex)e = (eye)(exe) = ee = e$$

If  $x, y, z \in T_1 \cup T_2$ , then using  $e(xy)e = e$ ,

$$xyz = (xex)(yeey)(zez) = x(exye)(eyze)z = xeez = x(exze)z = (xex)(zez) = xz$$

Hence by the maximality of  $T_1, T_2$  we have  $T_1 = T_1 \cup T_2 = T_2$ . □

Let the equivalence relation on  $S$  corresponding to the partition in Lemma 2.2 be denoted by  $\sim$ , and let the equivalence class containing an element  $x$  of  $S$  be denoted by  $T_x$ . Thus for  $x, y \in S$ ,  $T_x = T_y \Leftrightarrow x \sim y \Leftrightarrow$  [both  $x$  and  $y$  belong to some maximal element of  $\mathcal{S}$ ].

**Lemma 2.3.** *For all  $x, y \in S$ ,  $T_x = T_y \Leftrightarrow [xyx = x \text{ and } yxy = y]$ .*

*Proof.* One direction is trivial. Suppose now that  $xyx = x$  and  $yxy = y$ . Then the set  $\{x, y\}$  satisfies the identity  $xyz = xz$ , so by Lemma 2.1 can be extended to a maximal element of  $\mathcal{S}$ . □

Lemma 2.3 shows that  $T_x = T_y \Leftrightarrow SxS \cup \{x\} = SyS \cup \{y\}$ , so that the sets  $T_x$  are the  $J$ -classes of  $S$ .

**Lemma 2.4.** *For all  $x, y \in S$ ,  $xy \sim yx$ . Furthermore,  $\sim$  is a congruence on  $S$ , that is, for all  $x, y, x', y' \in S$ , if  $x \sim x'$  and  $y \sim y'$  then  $xy \sim x'y'$ .*

*Proof.* Clearly  $(xy)(yx)(xy) = xy$  and  $(yx)(xy)(yx) = yx$ , hence by Lemma 2.3,  $xy \sim yx$ . Now assume that  $x \sim x'$  and  $y \sim y'$ . Then

$$xy = (xx'x)(yy'y) \sim (x'xx')(y'yy') = x'y'$$

□

Let  $Q(S) = S / \sim$ . The elements of  $Q(S)$  are the sets  $T_x$ ,  $x \in S$ , and the multiplication is defined by  $T_x T_y = T_{xy}$ . In the standard terminology, Lemma 2.3 shows that each  $T_x$  is a rectangular idempotent semigroup ( $xyx = x$  for all  $x, y$ ), and Lemma 2.4 shows that  $Q(S)$  is a semilattice (commutative idempotent subgroup). (As remarked by McLean [6], every rectangular idempotent semigroup is “anticommutative” in the sense that  $xy = yx \Rightarrow x = xyx = yxy = y$ . Conversely, every anticommutative idempotent semigroup is rectangular:  $x \cdot xyx = yxy \cdot x \Rightarrow x = xyx$ .)

Let  $S$  be generated by  $g_1, g_2, \dots, g_n$ , and let  $x \in S$ . The *content* of  $x$  is the set  $C(x) = \{g_i : x = ag_i b \text{ for some } a, b \in S\}$ . The *reduced form* of  $x$  is  $\pi(x) = g_{i_1} g_{i_2} \cdots g_{i_k}$ , where  $C(x) = \{g_{i_1}, g_{i_2}, \dots, g_{i_k}\}$  and  $i_1 < i_2 < \cdots < i_k$ . (Of course these definitions depend on the set of generators.)

**Lemma 2.5.** *For all  $x \in S$ ,  $x \sim \pi(x)$ .*

*Proof.* This follows immediately from 2.4. □

According to Lemma 2.5,  $Q(S)$  is isomorphic to a subsemigroup of the semigroup of all non-empty subsets of  $\{1, 2, \dots, n\}$ , with set union as the operation.

**Theorem 2.1.** For all  $n \geq 1$ , every idempotent semigroup  $S$  on  $n$  generators is finite.

*Proof.* Let  $S$  have generators  $g_1, g_2, \dots, g_n$ . The proof has just two ingredients. The first is the fact that if  $C(x) = C(z) = \{g_1, g_2, \dots, g_n\}$ , then for any  $y \in S$ ,  $C(x) = C(yz) = C(z)$ , so that  $x, yz, z \in T_x$  and  $xyz = x(yz)z = xz$ . The second ingredient is a natural definition of the *length* of an element of  $S$ . The length of an element of  $S$  is defined below, and the proof of the theorem is then identical with the proof below.  $\square$

### 3 The Second Proof

Let  $S$  be an idempotent semigroup generated by  $g_1, g_2, \dots, g_n$ . Let us call an element  $x \in S$  *complete* if for each  $i$ ,  $1 \leq i \leq k$ , there are elements  $a_i$  and  $b_i$  of  $S$  such that  $x = a_i g_i b_i$ . For example,  $x = g_1 g_2 \cdots g_n$  is complete. For each  $x \in S$ , the *length* of  $x$ , denoted by  $|x|$ , is the minimum  $k$  such that  $x = x_1 x_2 \cdots x_k$ , where  $x_i \in \{g_1, g_2, \dots, g_n\}$ ,  $1 \leq i \leq k$ . Note that  $|x| \geq 1$  for all  $x \in S$ .

**Lemma 3.1.** If  $w \in S$  and  $w$  is complete, then  $w = wxw$  for all  $x \in S$ .

*Proof.* Let  $w \in S$  be complete. We show that  $w = wxw$  for all  $x \in S$  by induction on  $|x|$ . If  $|x| = 1$  then  $w = axb$  since  $w$  is complete, and

$$\begin{aligned} w &= (ax)b = (axax)b \\ &= a(xaxb) = a(xaxbxaxb) \\ &= (axax)bxaxb = (ax)bxaxb \\ &= (axb)x(axb) = wxw \end{aligned}$$

For the induction step, let  $|x| > 1$  and assume that  $w = wyw$  for all  $y \in S$  with  $|y| < |x|$ . Let  $x = yz$ , where  $|y| < |x|$  and  $|z| < |x|$ . Then  $w = wyw$  and  $w = wzw$ , so

$$\begin{aligned} w &= ww = (wzw)wyw = wzwyw \\ &= w(zwy)w = w(zwyzwy)w \\ &= (wzw)yz(wyw) = wyzw = wxw \end{aligned}$$

$\square$

**Lemma 3.2.** If  $x, y, z \in S_n$ , and  $x, z$  are complete, then  $xyz = xz$ .

*Proof.* Using Lemma 3.1,  $xyz = xy(zxz) = (xyzx)z = xz$ .  $\square$

**Theorem 3.1.** For each  $n \geq 1$ , every idempotent semigroup  $S$  with  $n$  generators is finite.

*Proof.* We will show by induction on  $n$  that for each  $n \geq 1$ , there is a finite upper bound  $t_n$  for the largest possible length of an element of any idempotent semigroup with  $n$  generators. Clearly  $t_1 = 1$  and  $t_2 = 3$ . Let  $n \geq 3$  and assume that  $t_{n-1}$  exists. Suppose that some element  $w$  of an idempotent semigroup  $S$  with  $n$  generators has length  $|w| > 2(t_{n-1} + 1)$ . Write  $w = xyz$ , where  $|x| = |z| = t_{n-1} + 1$ . By the induction hypothesis,  $x$  and  $z$  must be complete, i.e. each of the  $n$  generators of  $S$  occurs in each of  $x$  and

z. By Lemma 3.2,  $w = xyz = xz$ , so  $|w| \leq |x| + |z| = 2(t_{n+1} + 1)$ , a contradiction. Hence  $t_n$  exists and  $t_n \leq 2(t_{n-1} + 1)$ .  $\square$

Department of Mathematics  
Simon Fraser University  
Burnaby, BC, V6G 1G4  
Canada  
tbrown@sfu.ca

Department of Mathematics  
Southern Illinois University  
Edwardsville, IL 62026  
USA  
elazerson@sbcglobal.net

## References

- [1] T.C. Brown, *On the finiteness of semigroups in which  $x^r = r$* , Proc. Cambridge Philos. Soc. **60** (1964), 1028–1029.
- [2] J.A. Green and D. Rees, *On semi-groups in which  $x^r = x$* , Proc. Cambridge Philos. Soc. **48** (1952), 35–40.
- [3] Gérard Lallement, *Semigroups and combinatorial applications*, Pure and Applied Mathematics. A Wiley-interscience Publication, John Wiley & Sons, New York-Chichester-Brisbane, 1979.
- [4] Earl Lazerson, *Idempotent semigroups*, Working Paper No. 38, Communications Research Division, Institute for Defense Analyses, von Neumann Hall, Princeton, NJ, 1961.
- [5] M. Lothaire (ed.), *Combinatorics on words*, Cambridge University Press, Cambridge Mathematical Library, Cambridge, 1997, Corrected reprint of the 1983 original, with a new preface by Perrin.
- [6] David McLean, *Idempotent semigroups*, Amer. Math. Monthly **61** (1954), 110–113.