Affine and Combinatorial Binary m-Spaces

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Abstract

Let $V(n)$ denote the n-dimensional vector space over the 2-element field. Let $a(m,r)$ (respectively, $c(m,r)$) denote the smallest positive integer such that if $n \geq a(m,r)$ (respectively $n \geq c(m,r)$), and $V(n)$ is arbitrarily partitioned into r classes C_i, $1 \leq i \leq r$, then some class C_i must contain an m-dimensional affine (respectively, combinatorial) subspace of $V(n)$. Upper bounds for the functions $a(m,r)$ and $c(m,r)$ are investigated, as are upper bounds for the corresponding "density functions" $\bar{a}(m,\varepsilon)$ and $\bar{c}(m,\varepsilon)$.

1 Introduction and definitions

Throughout, $V(n)$ denotes the n-dimensional vector space over the 2-element field $\mathbb{F}_2 = \{0,1\}$:

$$V(n) = \{(x_1,\ldots,x_n) : x_i \in \mathbb{F}_2, 1 \leq i \leq n\}.$$

For integers $m \geq 1$, $r \geq 1$, $a(m,r)$ is defined to be the smallest positive integer such that if $n \geq a(m,r)$ and $V(n)$ is arbitrarily partitioned into r classes C_i, $1 \leq i \leq r$, then some class C_i contains an affine m-space. (An *affine m-space* is any translate (coset) of an m-dimensional vector subspace of $V(n)$. An affine 1-space is usually called an *affine line*.)

Similarly, $c(m,r)$ is defined to be the smallest positive integer such that if $n \geq c(m,r)$ and $V(n)$ is arbitrarily partitioned into r classes then some class must contain a combinatorial m-space. (The definition of combinatorial m-space is given in Section 3 below.)

The existence of $a(m,r)$ and $c(m,r)$ for all m,r is a consequence of a special case of the extended Hales-Jewett theorem [5,6].

In this note we investigate upper bounds for the functions $a(m,r)$ and $c(m,r)$.

We also inverstigate upper bounds for the corresponding density functions $\bar{a}(m,\varepsilon)$ and $\bar{c}(m,\varepsilon)$, which are defined as follows.

For any integer $m \geq 1$ and real number $\varepsilon > 0$, $\bar{a}(m,\varepsilon)$ (respectively, $\bar{c}(m,\varepsilon)$) is defined to be the smallest integer such that if $n \geq \bar{a}(m,\varepsilon)$ (respectively, $n \geq \bar{c}(m,\varepsilon)$) and A is an arbitrary subset of $V(n)$ which contains at least $\varepsilon|V(n)|$ elements, then A must contain an affine (respectively, combinatorial) m-space.

The existence of $\bar{a}(m,\varepsilon)$ follows from a result of Brown and Buhler [2, Lemma 1], which in turn is based upon a lemma of Szemerédi [7]. (See also Graham, Rothschild, and Spencer [5, p. 44] and Graham [4, p. 19].)

The existence of $\bar{c}(m,\varepsilon)$ is a consequence of a different result of Brown and Buhler [3]. (The existence of $\bar{a}(m,\varepsilon)$ also folows from this latter result.)

2 Upper bounds for $a(m,r)$ and $\bar{a}(m,\varepsilon)$

For the definitions of $a(m,r)$ and $\bar{a}(m,\varepsilon)$, see Section 1.

Theorem 1. *For $m \geq 1$, $k \geq 1$,*

$$\bar{a}(m,2^{-k}) \leq 2^m(k+2). \tag{1}$$

Proof. Let $m \geq 1$, $k \geq 1$ be given, and let $n = 2^m(k+2)$.

Let $V = V(n)$, so that $|V| = 2^n$ and $n = \log|V|$. (All logarithms here are taken with base 2.)

Now let $\varepsilon = 2^{-k}$, and let

$$A \subset V, \quad |A| \geq \varepsilon|V|.$$

One obtains, after a little juggling,

$$m = \log\log|V| - \log\log(4/\varepsilon).$$

It is shown in [2] that under exactly these circumstances the subset A must contain an affine m-space. Therefore $\bar{a}(m,2^{-k}) \leq n = 2^m(k+2)$, as required. □

Remark 1. When ε is not 2^{-k}, one can still use Theorem 1 to get

$$\bar{a}(m,\varepsilon) \leq \bar{a}(m,2^{-k}) \leq 2^m(k+2),$$

where

$$2^{-k} < \varepsilon < 2^{-(k-1)}.$$

Theorem 2. *For $m \geq 1, k \geq 1$,*

$$a(m,2^k) \leq 2^m(k+2). \tag{2}$$

Proof. This follows immediately from Theorem 1, since if $V(n)$ is partitioned into 2^k classes, then at least one of these classes has density at least 2^{-k}. That is, if

$$V(n) = C_1 \cup \cdots \cup C_{2^k},$$

then for some i,

$$|C_i| \geq 2^{-k}|V(n)|.$$

Hence $a(m,2^k) \leq \bar{a}(m,2^{-k}) \leq 2^m(k+2)$. □

Remark 2. If r is not 2^k, then one obtains

$$a(m,r) \leq a(m,2^k) \leq 2^m(k+2)$$

where

$$2^{k-1} < r < 2^k.$$

Theorem 3.

$$a(m,1) = m, \qquad m \geq 1, \tag{3}$$

$$a(1,r) = 1 + [\log_2 r], \qquad r \geq 1, \tag{4}$$

$$a(2,2^k - 1) \leq 3k, \qquad k \geq 2, \tag{5}$$

$$a(3,2^k - 1) \leq 10k - 2, \qquad k \geq 3, \tag{6}$$

$$a(3,3) \leq 15, \qquad a(4,3) \leq 55. \tag{7}$$

Proof. Equalities (3) and (4) are obvious. Note that any 2-element suset of $V(n)$ is an affine line in $V(n)$.

Inequalities (5) and (6) are proved using the following method.

To prove (5), fix $k \geq 2$ and let

$$V(3k) = V(2k) \times V(k)$$

be partitioned into $2^k - 1$ classes C_i, $1 \leq i \leq 2^k - 1$. We need to show that some affine 2-space in $V(3k)$ is contained in some C_i.

Let $y \in V(2k)$. Then $\{y\} \times V(k)$ is partitioned into $2^k - 1$ classes

$$(\{y\} \times V(k)) \cap C_i, \quad 1 \leq i \leq 2^k - 1.$$

Since $a(1,2^k - 1) = k$, there is an affine line

$$f(y) \subset V(k)$$

such that the affine line

$$\{y\} \times f(y) \subset V(3k)$$

is contained in some C_i.

Now we partition $V(2k)$ into $(2^k - 1)^2$ classes $D(i,j)$, $1 \leq i \leq 2^k - 1$, $1 \leq j \leq 2^k - 1$, in the following way. Let the distinct 1-dimensional vector subspaces of $V(k)$ be denoted by S_j, $1 \leq j \leq 2^k - 1$. Then the element y of $V(2k)$ belongs to the class $D(i,j)$ if and only if $\{y\} \times f(y) \subset C_i$ and $f(y)$ is a translate of S_j.

Since $a(1,(2^k - 1)^2) = 1 + [\log_2 (2^k - 1)^2] = 2k$, there is an affine line $\{y_1, y_2\}$ contained in some $D(i,j)$. It follows that

$$\{y_1\} \times f(y_1) \cup \{y_2\} \times f(y_2)$$

is an affine 2-space contained in C_i. This proves (5).

The proof of (6) uses the same idea. Fix $k \geq 3$, and let

$$V(10k-2) = V(7k-2) \times V(3k)$$

be partitioned into $2^k - 1$ classes C_i, $1 \leq i \leq 2^k - 1$. We need to show that some affine 3-space of $V(10k-2)$ is contained in some C_i.

Let $y \in V(7k-2)$; then $\{y\} \times V(3k)$ is partitioned into

$$(\{y\} \times V(3k)) \cap C_i, \quad 1 \leq i \leq 2^k - 1.$$

Since $a(2, 2^k - 1) \leq 3k$, there is an affine 2-space

$$f(y) \subset V(3k)$$

such that the affine 2-space

$$\{y\} \times f(y) \subset V(10k-2)$$

is contained in some C_i.

Now we partition $V(7k-2)$ into $(2^k - 1)t$ classes $D(i, j)$, $1 \leq i \leq 2^k - 1$, $1 \leq j \leq t$, where

$$t = \frac{(2^{3k} - 1)(2^{3k} - 2)}{(2^2 - 1)(2^2 - 2)}$$

is the number of 2-dimensional vector subspaces of $V(3k)$, just as before: Let the 2-dimensional vector subspaces of $V(3k)$ be denoted by S_j, $1 \leq j \leq t$. Then the element y of $V(7k-2)$ belongs to the class $D(i, j)$ if and only if $\{y\} \times f(y) \subset C_i$ and $f(y)$ is a translate of S_j.

Now $a(1, (2^k - 1)t) = 1 + [\log_2(2^k - 1)t] = 7k - 2$ (for this we need $k \geq 3$), and hence there is an affine line $\{y_1, y_2\}$ contained in some $D(i, j)$. It follows that

$$\{y_1\} \times f(y_1) \cup \{y_2\} \times f(y_2)$$

is an affine 3-space contained in C_i. This proves (6).

The bounds in (7) are proved using the same method. □

Remark 3. If the method above is continued to $a(4, 2^k - 1)$, $a(5, 2^k - 1)$ and so on, the resulting bounds are not as strong as those given by Theorem 2 (With the exception of $a(4, 3)$.)

3 Upper bounds for $c(m, r)$ and $\bar{c}(m, \varepsilon)$

For the definitions of $c(m, r)$ and $\bar{c}(m, \varepsilon)$, see Section 1.

Definition 1. A *combinatorial m-space* in $V(n)$ is any set S (of 2^m elements of $V(n)$) which can be described as follows. For some partition

$$\{1, 2, \ldots, n\} = B_0 \cup B_1 \cup \cdots \cup B_m,$$

where B_0 may be empty but B_j is not empty, $1 \le j \le m$, and for some function f from B_0 into $\mathbb{F}_2$, S is the set of all points $(x_1, \ldots, x_n)$ in $V(n)$ such that

$$x_i = f(i), \quad \text{for } i \in B_0,$$

$$x_i = x_{i'}, \quad \text{for } i, i' \in B_j, 1 \le j \le m.$$

A combinatorial 1-space is usually called a *combinatorial line*.

For example, with $m = 3$, $n = 8$, $B_0 = \{1,2\}$, $B_1 = \{3,4\}$, $B_2 = \{5\}$, $B_3 = \{6,7,8\}$, $f(1) = f(2) = 1$, S is the 3-space

$$\begin{array}{cccccccc} 11 & 00 & 0 & 000 & \quad 11 & 11 & 0 & 000 \\ 11 & 00 & 0 & 111 & \quad 11 & 11 & 0 & 111 \\ 11 & 00 & 1 & 000 & \quad 11 & 11 & 1 & 000 \\ 11 & 00 & 1 & 111 & \quad 11 & 11 & 1 & 111 \end{array}.$$

With $m = 1$, $n = 8$, $B_0 = \{1,2,3,4\}$, $B_1 = \{5,6,7,8\}$, $f(1) = f(4) = 1$, $f(2) = f(3) = 0$, S is the line

$$\begin{array}{c} 10010000 \\ 10011111 \end{array}.$$

Theorem 4. *For each $r \ge 1$,*

$$c(1,r) = r.$$

Proof. If $V(r) = C_1 \cup \cdots \cup C_r$, then some two of

$$\begin{array}{c} 000\cdots0 \\ 100\cdots0 \\ \vdots\ \vdots\ \vdots \quad \vdots \\ 111\cdots1 \end{array}$$

belong to the same class B_i. Hence $c(1,r) \le r$.

Let $V(r-1) = C_0 \cup \cdots \cup C_{r-1}$, where for $0 \le i \le r-1$,

$$C_i = \left\{ (x_1, \ldots, x_{r-1}) \in V(r-1) : \sum_{j=1}^{r-1} x_j = i \right\}.$$

Then no C_i contains a combinatorial line, Hence $c(1,r) \ge r$. □

The proof of Theorem 5 below is similar to the proof of Theorem 3. The main part of the proof is contained in the following Lemma.

Lemma 1. *Let $m \ge 1$, $r \ge 1$, be given, and let $t(m,r)$ be the number of distinct combinatorial m-spaces contained in $V(c(m,r))$. Then*

$$c(m+1,r) \le r \cdot t(m,r) + c(m,r).$$

Proof. For convenience write $t = t(m,r)$ and $s = c(m,r)$. Let

$$V(rt+s) = V(rt) \times V(s)$$

be partitioned into r classes C_i, $1 \leq i \leq r$. To show that some combinatorial $(m+1)$-space in $V(rt+s)$ is contained in some C_i, we proceed just as in the proof of Theorem 3.

For each $y \in V(rt)$, $\{y\} \times V(s)$ has been partitioned into classes

$$\{y\} \times V(s) \cap C_i, \quad 1 \leq i \leq r,$$

and since $s = c(m,r)$ there is a combinatorial m-space

$$f(y) \subset V(s)$$

such that the combinatorial m-space

$$\{y\} \cap f(y) \subset V(rt+s)$$

is contained in some C_i.

Now partition $V(rt)$ into rt classes $D(i,j)$, $1 \leq i \leq r$, $1 \leq j \leq t$, by putting the element y of $V(rt)$ into the class $D(i,j)$ if and only if $\{y\} \times f(y) \subset C_i$ and $f(y) = S_j$, where $S_1, \ldots, S_t$ are the combinatorial m-spaces in $V(s)$.

Since $c(1,rt) = rt$, there is a combinatorial line $\{y_1, y_2\}$ contained in some $D(i,j)$, and therefore

$$\{y_1\} \times f(y_1) \cup \{y_2\} \times f(y_2)$$

is a combinatorial $(m+1)$-space contained in C_i. □

Lemma 2. *The number of combinatorial m-spaces in $V(s)$ is*

$$\frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m+2-j)^s.$$

Proof. Each combinatorial m-space in $V(s)$ corresponds to an s-tuple on the $m+2$ symbols $0, 1, b_1, \ldots, b_m$, in which each b_j, $1 \leq j \leq m$, occurs at least once. In the examples described after the Definition above, the 3-space corresponds to $11b_1b_1b_2b_3b_3b_3$, and the 1-space (line) corresponds to $1001b_1b_1b_1b_1$. Counting by inclusion-exclusion gives the result. □

Theorem 5. *For $m = 2$, $r \geq 1$, we have*

$$c(2,r) < r \cdot (3^r - 2^r) + r < r \cdot 3^r.$$

In general, for $m \geq 2$, $r \geq 1$,

$$c(m,r) < r \cdot ((m+1)r)^{(m^r)^{\cdot^{\cdot^{(4^r)^{(3^r)}}}}}.$$

Proof. This is a crude estimate based on Lemmas 1 and 2. □

We now turn to upper bounds for the function $\bar{c}(m,\varepsilon)$.

Theorem 6. *For $m \geq 1$, $r \geq 1$,*

$$\bar{c}(m,1) = 1;$$
$$\bar{c}(1,1/r) \leq r^2.$$

Proof. It is an easy consequence of Sperner's lemma on families of pairwise incomparable subsets of a set (see [4] for details) that if A is any subset of $V(n)$ such that A contains no combinatorial line then

$$|A| \leq \binom{n}{[n/2]}.$$

Using

$$n^n e^{-n}\sqrt{2\pi n} e^{1/(12n+1)} \leq n! \leq n^n e^{-n}\sqrt{2\pi n} e^{1/12n},$$

one obtains

$$\binom{n}{[n/2]} < \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}} \cdot 2^n, \quad n \geq 1.$$

Hence if $A \subset V(r^2)$, $|A| \geq (1/r)|V(r^2)|$, then

$$|A| \geq \frac{1}{r} \cdot 2^{r^2} > \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{r^2}} \cdot 2^{r^2} > \binom{r^2}{[r^2/2]},$$

and therefore A contains a combinatorial line. □

Lemma 3. *For $m \geq 1$, $r \geq 1$, let $n = \bar{c}(m, 1/(r+1))$ and let e be the number of distinct combinatorial m-spaces contained in $V(n)$. Then*

$$\bar{c}(m+1,1/r) \leq n + r^4 e^2.$$

In particular, using Theorem 6 and Lemma 2,

$$\bar{c}(2,1/r) \leq (r+1)^2 + r^4(3^{(r+1)^2} - 2^{(r+1)^2})^2.$$

Proof. It is shown in [3] that

$$\bar{c}(m+1,1/r) \leq n + \bar{c}(1,(r^2 e)^{-1}).$$

Applying Theorem 6 gives the result. □

One can now apply Lemma 3 (and Lemma 2) repeatedly to get an explicit upper bound for $\bar{c}(m,1/r)$. One estimate obtained in this way is the following.

Theorem 7. *For $m \geq 2$, $r \geq 1$, let $s = (r+m-1)^4$. Then*

$$c(m,1/r) \leq r^4((m+1)^s)^{(m^s)^{\cdot^{\cdot^{(4^s)^{(3^s)}}}}}.$$

4 Remarks

Since Theorem 1 above is surely *much* stronger than Theorem 2, it should be possible to considerably strengthen Theorem 2.

Replacing the two-element field $\mathbb{F}_2$ by the three-element field $\mathbb{F}_3$ (or any larger field) leads to considerable difficulties. Let $a(m,r,q),\dots$ be the functions defined analogously to $a(m,r),\dots$, where $\mathbb{F}_2$ is replaced by the q-element field $\mathbb{F}_q$. (The definitions of affine m-space and combinatorial m-space remain unchanged. Note, however, that the definition of a combinatorial m-space does not require a finite field, but only a finite set.)

The existence of $a(m,r,q)$ and $c(m,r,q)$ follows from the Hales-Jewett theorem. The existence of $\bar{a}(m,\varepsilon,q)$ is known only for $q=2$ and $q=3$ [2, 3], and the existence of $\bar{c}(m,\varepsilon,q)$ is not known even for $q=3$. R. L. Graham has offered a reward [4] for an answer to the question of the existence of $\bar{c}(1,\varepsilon,3)$.

Following the methods used above to prove Theorems 3, 5, and 7, one could calculate upper bounds for, say, $a(m,r,3)$ and $\bar{a}(m,\varepsilon,3)$, and $c(m,r,3)$ in terms of upper bounds for the case $m=1$. However, no satisfactory upper bounds for $a(1,r,3)$, $\bar{a}(1,\varepsilon,3)$, and $c(1,r,3)$ have been found.

(It is trivial that $a(1,1,3)=1$ and $a(1,2,3)=2$. A recent calculation [1] shows that $a(1,3,3)=4$.)

Finally, we remark that by identifying $V(n)$ with the set of subsets of $\{1,\dots,n\}$ in the natural way, the combinatorial m-spaces in $V(n)$ become identified with collections of the form

$$\left\{A_0 \cup \bigcup_{i\in I} A_i : I \subset \{1,\dots,n\}\right\},$$

where $A_0, A_1, \dots, A_m$ are pairwise disjoint subsets of $\{1,\dots,n\}$ and $A_1,\dots,A_m$ are non-empty.

The functions $c(m,r)$ and $\bar{c}(m,1/r)$ can be interpreted from this point of view. Related bounds have been found by Alan Taylor [8] for the case where A_0 is empty and I is non-empty.

References

[1] T.C. Brown, *Monochromatic affine lines in finite vector spaces*, J. Combin. Theory Ser. A **38** (1985), 35–41.

[2] T.C. Brown and J.P. Buhler, *A density version of a geometric Ramsey theorem*, J. Combin. Theory Ser. A **25** (1982), 20–34.

[3] ———, *Lines imply spaces in density Ramsey theory*, J. Combin. Theory Ser. A **36** (1984), 214–220.

[4] Ron L. Graham, *Rudiments of ramsey theory*, Amer. Math. Soc., Providence, RI, 1981.

[5] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, *Ramsey theory*, Wiley-Interscience Series in Discrete Mathematics. A Wiley-Interscience Publication., John Wiley & Sons, Inc., New York, 1980.

[6] Alfred W. Hales and Robert I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222–239.

[7] E. Szemerédi, *On sets of integers containing no k elements in an arithmetic progression*, Acta. Arith. **27** (1975), 199–245, Collection of articles in memory of Jurii Vladimirovic Linnik.

[8] Alan D. Taylor, *Bounds for the disjoint unions theorem*, J. Combin. Theory Ser. A **30** (1981), 339–344.