Lines Imply Spaces in Density Ramsey Theory

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Abstract

Some results of geometric Ramsey theory assert that if $F$ is a finite field (respectively, set) and $n$ is sufficiently large, then in any coloring of the points of $F^n$ there is a monochromatic $k$-dimensional affine (respectively, combinatorial) subspace (see [9]). We prove that the density version of this result for lines (i.e., $k = 1$) implies the density version for arbitrary $k$. By using results in [2, 6] we obtain various consequences: a “group-theoretic” version of Roth’s Theorem, a proof of the density assertion for arbitrary $k$ in the finite field case when $|F| = 3$, and a proof of the density assertion for arbitrary $k$ in the combinatorial case when $|F| = 2$.

1 Results

In this section we will state and discuss the main results and prove some corollaries. The proofs of the main results are in the following section. Throughout $q$ denotes a prime power.

Let $\mathbb{F}_q$ be the field with $q$ elements and let $V$ be an $n$-dimensional vector space over $\mathbb{F}_q$. For each positive integer $k$ and positive real number $\varepsilon$ let $n(\varepsilon, k, q)$ denote the smallest integer (if one exists) such that

$$n = \dim_{\mathbb{F}_q} V \geq n(\varepsilon, k, q), \quad A \subset V, \quad |A| > \varepsilon|V|,$$

imply that $A$ contains an affine $k$-space. (By an affine $k$-space we mean any translate of a $k$-dimensional vector subspace; the purist will note that we only use the structure of $V$ as an affine space.)

The “Affine Line Conjecture” is the assertion that $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$ and all $q$. The existence of $n(\varepsilon, k, q)$ would be a density version of the results in [9] on Ramsey theorems in geometric contexts.

The main assertion of this paper is that if, for a fixed $q$, $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$, then $n(\varepsilon, k, q)$ exists for all $k$ and all $\varepsilon > 0$. We will also reinterpret this result in the context of “combinatorial” $k$-spaces and “lattices” in abelian groups. We include a number of corollaries and remarks.

(It is not hard to see that if $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$ and all $q$, then $n(\varepsilon, k, q)$ exists for all $k$, $\varepsilon$, and $q$. Indeed, if $\varepsilon, k$, and $q$ are given, let $F$ be the extension of $\mathbb{F}_q$ of degree $k$. An affine line in an $F$-vector space is a $k$-space over $\mathbb{F}_q$ if we “restrict scalars” to $\mathbb{F}_q$; from this it is easy to see that the existence of an affine line in a large enough subset of $F^n$ implies the existence of an affine $k$-space in a large enough subset of $\mathbb{F}_q^{kn}$.)
Theorem 1. Suppose that $\mathbb{F}_q$ is a fixed finite field and that $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$. Then $n(\varepsilon, k, q)$ exists for all $\varepsilon > 0$ and all $k$.

Corollary. The integers $n(\varepsilon, k, 2)$ and $n(\varepsilon, k, 3)$ exist for all $\varepsilon > 0$ and all $k$.

Proof of the corollary. Any two-element subset of an $\mathbb{F}_2$ vector space is an affine line so it is trivial that $n(\varepsilon, 1, q)$ exists. The theorem then implies that $n(\varepsilon, k, 2)$ exists for all $k$ (see the corollary to Lemma 1 in [2] for a different proof of the existence of $n(\varepsilon, k, 2)$). The existence of $n(\varepsilon, k, 3)$ follows from Theorem 1 and the existence of $n(\varepsilon, 1, 3)$ which is the central result of [2]. This finishes the proof of the corollary.

A set $\{x_1, \ldots, x_k\}$ of the elements in an abelian group $G$ is said to be independent if $c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0$ implies that $c_i x_i = 0$ for each $i$. An $(m, k)$-lattice in an abelian group $G$ is a set of the form

$$M = \{a + c_1 x_1 + \cdots + c_k x_k : c_i = 0, 1, \ldots, m-1\},$$

where $a$ is an element of $G$ and the $x_i$ are independent. If $V$ is a vector space over a finite field, then by an $(m, k)$-lattice in $V$ we mean an $(m, k)$-lattice in its underlying additive group.

Let $n'(\varepsilon, k, q)$ denote the smallest integer (if one exists) such that if

$$n = \dim_{\mathbb{F}_q} V \geq n'(\varepsilon, k, q), \quad A \subset V, \quad |A| > \varepsilon|V|,$$

then $A$ contains a $(3, k)$-lattice.

Theorem 2. $n'(\varepsilon, k, q)$ exists for all $\varepsilon > 0$, $k$, and $q$.

Corollary. For each $\varepsilon > 0$ and positive integer $k$ there is an integer $m(\varepsilon, k)$ such that if $G$ is any finite abelian group with more than $m(\varepsilon, k)$ elements and $A$ is any subset of $G$ with more that $\varepsilon |G|$ elements, then there is a $(3, k)$-lattice inside $A$.

Proof of the corollary. Let $k$ and $\varepsilon$ be given. Choose by Szemerédi’s theorem [10] a large enough $n$ so that any subset of $\{1, 2, \ldots, n\}$ with more than $\varepsilon n$ elements contains an arithmetic progression with $3^k$ terms. Choose $m(\varepsilon, k)$ large enough so that any finite abelian group $G$ with more than $m(\varepsilon, k)$ elements must contain either a cyclic subgroup $H$ of order at least $n$, or a subgroup $H$ which is the direct product of at least $n'(\varepsilon, k, p)$ cyclic groups of order $p$ for some prime $p < n$.

Now let $G$ be a finite abelian group with more than $m(\varepsilon, k)$ elements and let $A$ be a subset of $G$ with $|A| > \varepsilon |G|$. Let $H$ be the subgroup whose existence is guaranteed by the choice of $m(\varepsilon, k)$. Then $|A \cap a + H| > \varepsilon |H|$ for some coset $a + H$ of $H$. If $H$ is cyclic, then $A - a$ contains the set

$$\{a_0 + c_1 d + c_2 (3d) + \cdots + c_k (3^{k-1}d) : c_i = 0, 1, 2\},$$

where $d$ is the difference of the arithmetic progression whose existence is guaranteed by the choice of $n$ above. If $H$ is the direct product of at least $n'(\varepsilon, k, q)$ cyclic groups of order $p$, then $A - a$ contains

$$\{a_0 + c_1 x_1 + \cdots + c_k x_k : c_i = 0, 1, 2\}$$

for an independent set of $x_i$. Thus in either case $A$ contains a $(3, k)$-lattice and we are finished.
Remarks. (1) Roth’s special case of Szemerédi’s theorem asserts that if \( n \) is sufficiently large and \( A \) is a subset of \( \{1, 2, \ldots, n\} \) with more than \( \epsilon n \) elements then \( A \) contains a set of the form \( \{a, a+\lambda, a+2\lambda\} \).

This is equivalent to the case \( k = 1 \) of the corollary in the case in which \( G \) is cyclic. Indeed, it is not hard to check that one has

\[
m(\epsilon, 1) \leq n \leq \frac{1}{2} m\left(\frac{\epsilon}{2}, 1\right) + 1
\]

(to verify the second inequality consider subsets of the “first half” of a sufficiently large cyclic group).

Thus the corollary could be thought of as a group-theoretic generalization of Roth’s Theorem.

(2) Since sufficiently large groups contain large abelian subgroups [4], we could actually delete the requirement that \( G \) be abelian in the statement of the corollary.

(3) If the Affine Line Conjecture is valid, then the results here imply the obvious “group-theoretic generalization” of Szemerédi’s Theorem: For every \( \epsilon > 0 \), \( k \), and \( \lambda \) there exists an integer \( m(\epsilon, k, \lambda) \) such that if \( G \) is any finite abelian group with more than \( m(\epsilon, k, \lambda) \) elements and \( A \) is any subset of \( G \) with more \( \epsilon|G| \) elements, then there exists an \((l, k)\)-lattice in \( A \).

Finally, we remove the algebraic structure on the underlying set, replacing \( \mathbb{F}_q \) with an arbitrary finite set. Thus we consider combinatorial subspaces; we briefly recall the definition (see [6] for further details).

Let \( F \) be the finite set \( \{0, 1, \ldots, t-1\} \) with \( t \) elements. A subset \( W \) of \( F^n \) is a combinatorial \( k \)-space if it satisfies the following. There is a partition

\[
\{1, \ldots, n\} = B_0 \cup B_1 \cup \cdots \cup B_k
\]

such that \( B_1, \ldots, B_k \) are nonempty. There is a function \( f : B_0 \to F \). A function \( \vec{f} : F^k \to F^n \) is defined by

\[
\vec{f}(y_1, \ldots, y_k) = (x_1, \ldots, x_n)
\]

where

\[
x_i = f(i) \quad \text{for } i \in B_0,
\]

\[
x_i = y_j \quad \text{for } i \in B_j, 1 \leq j \leq k.
\]

\( W \) is the range of \( \vec{f} \).

The definition is complicated, but it captures a notion of subspace when the only structure on \( F \) is that of a finite set. We remark that the Hales-Jewett Theorem [6, 7] asserts that if \( n \) is large enough, then in any coloring of \( F^n \) there is a monochromatic combinatorial 1-space (usually called a combinatorial line).

Let \( n''(\epsilon, k, t) \) be the smallest integer (if one exists) such that if

\[
n \geq n''(\epsilon, k, t), \quad A \subset F^n, \quad |A| > \epsilon|F^n|,
\]

then \( A \) contains a combinatorial \( k \)-space.

**Theorem 3.** Let \( t \) be fixed. If \( n''(\epsilon, 1, t) \) exists for all \( \epsilon > 0 \), then \( n''(\epsilon, k, t) \) exists for all \( \epsilon > 0 \) and all \( k \).

**Corollary.** \( n''(\epsilon, k, 2) \) exists for all \( \epsilon > 0 \) and all \( k \).

**Proof of the corollary.** The existence of \( n''(\epsilon, 1, 2) \) is a simple consequence of Sperner’s Lemma (see [1] or [6]).
Remarks. (1) In [1] it is shown that if there is a fixed $\varepsilon_0 < 1$ such that $n''(\varepsilon_0, 1, t)$ exists for all $t$, then $n''(\varepsilon, 1, t)$ exists for all $\varepsilon > 0$ and all $t$. The corresponding result for $n(\varepsilon, 1, q)$ is proved in [3].

(2) The existence of $n''(\varepsilon, 1, t)$ is a “density version” of the Hales-Jewett Theorem. Graham has offered a reward for a proof of the existence (or non-existence!) of the numbers $n''(\varepsilon, 1, 3)$.

2 Proofs

The following lemma contains the crucial idea underlying Theorems 1, 2, and 3.

Lemma. Let $\mathbb{F}_q$ be a fixed finite field and $k$ a fixed positive integer. Assume that $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$. Then for each positive integer $r$, if $n(1/(r+1), k, q)$ exists then $n(1/r, k+1, q)$ exists. Similar statements holds for $n'(\varepsilon, k, q)$ and $n''(\varepsilon, k, t)$.

Proof. We give the proof in the vector space case $n(\varepsilon, k, q)$. The proofs for $n'(\varepsilon, k, q)$ and $n''(\varepsilon, k, t)$ are entirely analogous. In the lattice case $n'(\varepsilon, k, q)$ it is merely necessary to replace “$k$-space” with “$(3, k)$-lattice” and “line” with “$(3, 1)$-lattice” throughout. In the combinatorial case $n''(\varepsilon, k, t)$ it is necessary to replace “affine $k$-space” with “combinatorial $k$-space” and “affine line” with “combinatorial line” throughout.

Let $n_0 = n(1/(r+1), k, q)$. Let $e$ be the number of distinct $k$-dimensional vector subspaces of any $n_0$-dimensional vector space over $\mathbb{F}_q$. Let $\delta = (q^{n_0}e^r)^{-1}$ and let $s = n(\delta, 1, q)$. We claim that

$$n(1/r, k+1, q) \leq n_0 + s.$$ 

To prove this we must start with a vector space $V$ over $\mathbb{F}_q$ of dimension at least $n_0 + s$. Let $A$ be a subset of $V$ with

$$|A| > (1/r)|V| \geq (1/r)q^{n_0+s}.$$ 

Let $W_0$ be a $n_0$-dimensional subspace of $V$ and let

$$V = \bigcup W_\alpha$$

be the decomposition of $V$ into a union of the pairwise disjoint translates (cosets) of $W_0$. For the proof to work in the combinatorial case it is necessary at this point to choose $W_0$ to be the subspace consisting of the vectors whose last $s$ components are 0.

Let $t$ be the number of cosets $W_\alpha$ such that

$$|A \cap W_\alpha| \leq \frac{1}{r+1}|W_\alpha| = \frac{1}{r+1}q^{n_0}.$$ 

There are $q^t$ cosets altogether, so

$$\frac{1}{r}|V| < |A| = \sum |A \cap W_\alpha| \leq \frac{t}{r+1}|W_\alpha| + (q^t - t)|W_\alpha|.$$ 

This gives

$$q^t - t > q^t/r.$$
Hence there are \( d = q^t - t > q^t/r^2 \) cosets \( W_\alpha \) such that

\[
|A \cap W_\alpha| > \frac{1}{r+1} |W_\alpha|,
\]

and since the dimension of \( W_0 \) is \( n_0 = n(1/(r+1), k, q) \) each such \( A \cap W_\alpha \) must contain an affine \( k \)-space

\[
a_\alpha + U_\alpha,
\]

where \( U_\alpha \) is a \( k \)-dimensional vector subspace of \( W_\alpha \).

Since there are exactly \( e \) distinct \( k \)-dimensional vector subspaces of \( W_0 \) at least \( d/e \) of the \( k \)-spaces \( a_\alpha + U_\alpha \) must have the form \( a_\alpha + U \) for a fixed \( U \). Let these be

\[
a_1 + U, \ldots, a_h + U,
\]

where \( h \geq d/e \).

Let \( A' = \{ a_1, \ldots, a_h \} \). Then

\[
|A'| = h \geq d/e > \frac{q^s}{er^2} = \frac{1}{q^{rn_0+e}q^{rn_0+e}} = \delta |V|.
\]

Since the dimension of \( V \) is \( n_0 + s > s = n(\delta, 1, q) \), there must be an affine line in \( A' \) by renumbering if necessary we can assume that this line is \( \{ a_1, \ldots, a_q \} \).

It is now easy to check that

\[
U' = (a_1 + U) \cup \cdots \cup (a_q + U)
\]

is an affine \( (k+1) \)-space contained in \( A \). Since \( A \) was an arbitrary subset of \( V \) with \( |A| > (1/r)|V| \) this shows that

\[
n(1/r, k+1, q) \leq n_0 + s = \dim_{\mathbb{F}_q}(V)
\]

as claimed. This finishes the proof of the lemma. \( \square \)

Theorem 1 now follows immediately from the lemma by induction. Indeed, we are given in the hypotheses of the theorem that \( n(\varepsilon, 1,q) \) exists for all \( \varepsilon > 0 \). If \( n(\varepsilon, k,q) \) exists for all \( \varepsilon \), then it exists for \( \varepsilon = 1/r \). By the lemma, \( n(\varepsilon, k+1, q) \) exists for all \( \varepsilon > 0 \). Theorem 1 now follows by induction on \( k \).

The proof of Theorem 3 is identical; we merely replace \( n(\varepsilon, k,q) \) with \( n'(\varepsilon, k, t) \).

To prove Theorem 2 for odd \( q \) we first observe that \( n'(\varepsilon, 1,q) \) exists for all \( \varepsilon > 0 \) as a consequence of the main result in [2]. For this case Theorem 2 follows from the lemma and induction as above.

To prove Theorem 2 for even \( q \) we observe that a \((3,k)\)-lattice is just a \((2,k)\)-lattice since \( 2 = 0 \) in \( \mathbb{F}_q \). It then follows that \( n'(\varepsilon, 1,q) \) exists since any two elements of an abelian group form a \((2,1)\)-lattice. The rest of the proof is as above. (An upper bound for \( n'(\varepsilon, k,q) \) for even \( q \) can also be deduced from Lemma 1 in [2].)

Note added in proof. The lemma can be easily improve to show that \( n(1/r, k+1, q) \leq n(1/(r+1), k,q) + n(1/(er^2), 1,q) \).
References


