Common Transversals for Partitions of a Finite Set

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Abstract

For $s \geq 2$, $t \geq 1$, let $A_1, \ldots, A_t$ be s-cell partitions of a finite set X. Assume that if $x, y \in X$, $x \neq y$, then x, y belong to different cells of at least one of the partitions A_i. For each $k \geq 1$, let $c(s,t,k)$ be the least integer such that if $A_1, \ldots, A_t$, X satisfy the preceding conditions, and the smallest of all the cells of all the partitions has *exactly* k elements, and $|X| \geq c(s,t,k)$, then $A_1, \ldots, A_t$ have a common transversal. The functions $b(s,t,k)$ are defined analogously, except that now the smallest of all the cells of all the partitions is only required to have *at least* k elements. Thus $b(s,t,1)$ involves no restriction on the sizes of the cells of the partitions. Note that $b(s,t,1) = \max\{c(s,t,k) : k \geq 1\}$.

We show, using essentially the method of Longyear [4], that

(1) $c(s,t,1) = s^t - s^{t-1} - (s-1)^{t-1} + 2$, $s \geq 2$, $t \geq 1$, $(s,t) \neq (2,2)$;

(2) $c(s,3,s-1) \geq s^3 - s^2 - (s-1)^2 + s$, $s \geq 2$, $t = 3$;

(3) $c(s,t,(t-1)(s-1)^{t-2}) \geq (t-1)(t-2)(s-1)^{t-2} + s(s-1)^{t-1} + 1$, $s \geq 2$, $t \geq 3$, $s \geq t-2$;

(4) $b(s,2,[(s+1)/2]) = 0$, $s \geq 2$, $t = 2$;

(5) $c(s,t,s^k) \geq s^t - s^{t-1} - s^k(s-1)^{t-k-1} + s^k + 1$, $s \geq 2$, $k \geq 0$, $t \geq k+2$.

1 Introduction and definitions

The functions $b(s,t,k)$ (defined above) were introduced by Longyear in [4], who showed, among other results, that $b(s,2,1) = s^2 - 2s + 3$, $s \geq 3$.

The present author's primary interest is in the determination of the values of the function $b(s,t,1)$. In this note we give a number of partial results in this direction, mostly in the form of lower bounds obtained by various constructions. The exact values of $b(s,t,1)$, $s \geq 2$, however, are still unknown for all $t \geq 3$.

Throughout, $A_1, \ldots, A_t$ denote partitions of a finite set X, where each partition has s cells, $s \geq 2$. We also assume that the partitions $A_1, \ldots, A_t$ *separate points* of X in the sense that if $x, y \in X$, $x \neq y$, then x, y belong to different cells of at least one of the partitions A_i.

For each i, $1 \leq i \leq t$, we order the cells of A_i so that $A_i = (A(i,1), \ldots, A(i,s))$. Let $P(s,t)$ be the set of all t-tuples $a_1 \cdots a_t$, where each coordinate a_i is a member of $\{1, \ldots, s\}$, $1 \leq i \leq t$.

Now define a mapping g from X into $P(s,t)$ by setting, for each $x \in X$,

$$g(x) = a_1 \cdots a_t, \quad x \in A(1,a_1) \cap \cdots \cap A(t,a_t).$$

Then, since $A_1,\ldots,A_t$ separate the points of X, the mapping g is injective. Also, for $1 \le i \le t$, $1 \le j \le s$,

$$A(i,j) = \{x \in X : \text{the } i\text{th coordinate of } g(x) \text{ is } j\},$$

so that the partitions $A_1,\ldots,A_t$ can be recovered from $g(X)$.

Hence, from now on we shall identify X with $g(X)$, and work entirely with subsets of $P(s,t)$. This idea is due to Longyear.

Recall that a *transversal* of A_i is a set T of s elements of X, one element from each of the s cells of A_i. A *common transversal* of $A_1,\ldots,A_t$ is a set T of s elements of X such that T is a transversal of each A_i, $1 \le i \le t$.

A *complementary set* is a set D of s elements of $P(s,t)$ such that for each i, $1 \le i \le t$, the ith coordinates of the elements of D run through $\{1,\ldots,s\}$ in some order.

Note that the s-cell partitions $A_1,\ldots,A_t$ of X have a common transversal if and only if the subset $g(X)$ of $P(s,t)$ contains a complementary set. Hence the functions $c(s,t,k)$ and $b(s,t,k)$ defined above can be described as follows:

Let s,t be given, $s \ge 2$, $t \ge 1$. For a subset Q of $P(s,t)$ and any i, j, where $1 \le i \le t$, $1 \le j \le s$, let $q(a_i = j)$ be the number of elements of Q whose ith coordinate is j.

Then $c(s,t,k)$ is the smallest integer with the following property. If Q is any subset of $P(s,t)$ such that $|Q| \ge c(s,t,k)$ and such that $k = \min\{q(a_i = j) : 1 \le i \le t, 1 \le j \le s\}$, then Q contains a complementary set. Similarly, $b(s,t,k)$ is the smallest integer such that if Q is any subset of $P(s,t)$ such that $|Q| \ge b(s,t,k)$ and such that

$$k \le \min\{q(a_i = j) : 1 \le i \le t, 1 \le j \le s\},$$

then Q contains a complementary set.

Note that $c(s,t,k)$ is defined only for $k \ge 1$ (or else we may take $c(s,t,0) = |P(s,t)| + 1$), whereas $b(s,t,k)$ is defined for all $k \ge 0$.

Also note again that $b(s,t,k) \ge c(s,t,k)$ and that $b(s,t,1) = \max\{c(s,t,k) : k \ge 1\}$.

2 Results

The following lemma, which follows from simple counting, will be used repeatedly.

Lemma 1. *The set $P(s,t)$ contains $s!^{t-1}$ complementary sets altogether, each element of $P(s,t)$ belongs to $(s-1)!^{t-1}$ complementary sets, and each compatible pair of elements of $P(s,t)$ belongs to $(s-2)!^{t-1}$ complementary sets.*

Theorem 1. (Longyear [4]). *For $s \ge 2$, $t \ge 1$,*

$$b(s,t,0) = s^t - s^{t-1} + 1.$$

Proof 1. (Longyear [4]). Let Q be a subset of $P(s,t)$ with $|Q| \ge s^t - s^{t-1} + 1$. Let $B = P(s,t) - Q$. Then $|B| \le s^{t-1} - 1$, so by Lemma 1 B can intersect at most $|B|(s-1)!^{t-1}$ complementary sets, and hence Q

contains a complementary set. This shows that $b(s,t,0) \leq s^t - s^{t-1} + 1$. Now let $Q = P(s,t) - B$, where

$$B = \{a_1 \cdots a_t \in P(s,t) : a_1 = 1\}.$$

Then Q contains no complementary set, hence

$$b(s,t,0) \geq |Q| + 1 = s^t - s^{t-1} + 1. \qquad \square$$

Proof 2. We use induction on t, the case $t = 1$ being trivial. Assume the result for a given $t \geq 1$, and let Q be a subset of $P(s,t+1)$ with $|Q| \geq s^{t+1} - s^t + 1$. For each h, $0 \leq h \leq s-1$, let $B_h = \{a_1 \cdots a_t a_{t+1} \in Q : a_{t+1} - a_t \equiv h \pmod{s}\}$. Then Q is the disjoint union of the sets B_h, hence (re-numbering if necessary) we may assume that $|B_0| \geq s^t - s^{t-1} + 1$. Now let $Q' = \{a_1 \cdots a_t : a_1 \cdots a_t a_t \in B_0\}$. Since $|Q'| = |B_0|$, the induction hypothesis implies that Q' contains a complementary set D'. Then Q contains the complementary set $D = \{a_1 \cdots a_t a_t \in D'\}$. $\square$

Theorem 2. *For $s \geq 2$, $t \geq 1$, $(s,t) \neq (2,2)$,*

$$c(s,t,1) = s^t - s^{t-1} - (s-t)^{t-1} + 2.$$

Proof. When $t = 1$ there is nothing to prove. Hence assume that $t \geq 2$ and let Q be a subset of $P(s,t)$ such that $|Q| \geq s^t - s^{t-1} - (s-1)^{t-1} + 2$ and such that $1 = \min\{q(a_i = j) : 1 \leq i \leq t, 1 \leq j \leq s\}$, where, as before, $q(a_i = j)$ is the number of elements of Q whose ith coordinate is j. Let $B = P(s,t) - Q$, and assume without loss of generality that $1 = q(a_1 = 1)$, and that $B = B_1 \cup B_2$, where $B_1 = \{a_1 \cdots a_t \in P(s,t) : a_1 = 1\} - \{11 \cdots 1\}$ and $B_2 = B - B_1$. Now $|B| \leq s^{t-1} + (s-1)^{t-1} - 2$, and $|B_1| = s^{t-1} - 1$, so $|B_2| \leq (s-1)^{t-1} - 1$. The set B_1 meets every complementary set in $P(s,t)$ except for the $(s-1)!^{t-1}$ complementary sets which contain the t-tuple $11 \cdots 1$. The set B_2 can meet at most $|B_2|(s-2)!^{t-1}$ *of these*. (The complementary sets containing $11 \cdots 1$). Since $|B_2|(s-2)!^{t-1} < (s-1)1^{t-1}$, it follows that Q contains a complementary set, and hence that $c(s,t,1) \leq s^t - s^{t-1} - (s-1)^{t-1} + 2$.

For the reverse inequality let $Q = P(s,t) - (B_1 \cup B_2)$, where B_1 is as above and

$$B_2 = \{a_1 \cdots a_t \in P(s,t) : a_1 = 2, a_i \neq 1, 2 \leq i \leq t\}.$$

Then $B_1 \cup B_2$ meets every complementary set, hence Q contains no complementary set, and $1 = q(a_1 = 1) = \min\{q(a_i = j) : 1 \leq i \leq t, 1 \leq j \leq s\}$ (except in the single case $s = t = 2$, when $q(a_2 = 2) = 0$). Therefore $c(s,t,1) \geq |Q| + 1 = s^t - s^{t-1} - (s-1)^{t-1} + 2$, $s \geq 2$, $t \geq 1$, $(s,t) \neq (2,2)$. $\square$

Remark. For the case $t = 2$, Longyear showed using Hall's theorem [2] that $b(s,t,1) = s^2 - 2s + 3$, $s \geq 3$. Thus (checking the case $s = t = 2$ separately, where we find $b(2,2,1) = 2 = c(2,2,1)$)

$$b(s,2,1) = c(s,2,1), \quad s \geq 2.$$

Theorem 3. *For $s \geq 2$, $t = 3$,*

$$c(s,3,s-1) \geq s^3 - s^2 - (s-1)^2 + s.$$

Proof. Let Q be the set of all triple of the form $1a_2 1, a_1 11, a_1 a_2 b$, where $2 \le a_1 \le s$, $2 \le a_2 \le s$, $1 \le b \le s$. Then Q contains no complementary set and $s-1 = a(a_1 = 1) = \min\{q(a_i = j) : 1 \le i \le 3, 1 \le j \le s\}$, hence $c(s,3,s-1) \ge |Q| + 1 = 2(s-1) + (s-1)^2 s + 1 = s^3 - s^2 - (s-1)^2 + s$. □

Remark. Since $b(s,3,1) \ge c(s,3,s-1)$, and $s^3 - s^2 - (s-1)^s + s = c(s,3,1) + (s-2)$, Theorem 3 gives $b(s,3,1) \ge c(s,3,1) + (s-2)$. Thus $b(s,3,1) > c(s,3,1)$, $s > 2$. This is the only known case where $b(s,t,1) > c(s,t,1)$.

Theorem 4. *For* $s \ge 2$, $t \ge 3$, $s \ge t-2$,

$$c(s,t,(t-2)(s-2)^{t-2}) \ge (t-1)(t-2)(s-1)^{t-2} + s(s-1)^{t-1} + 1.$$

Proof. We generalize the contruction used in Theorem 3. Let Q be the set of all t-tuples of the form $1a_2 a_3 \cdots a_{t-1} b, a_1 1 a_3 \cdots a_{t-1} b, a_1 a_2 1 a_4 \cdots a_{t-1} b, \ldots, a_1 a_2 \cdots a_{t-2} 1 b, a_1 a_2 \cdots a_{t-2} a_{t-1} c$, where $2 \le a_1, \ldots, a_{t-1} \le s$, $1 \le b \le t-2$, $1 \le c \le s$. Then Q contains no complementary set and it is easy to check that

$$(t-2)(s-1)^{t-2} = q(a_1 = 1) = \min\{q(q_i = j) : 1 \le i \le t, 1 \le j \le s\},$$

hence

$$c(s,t,(t-2)(s-2)^{t-2}) \ge |Q| + 1 = (t-1)(t-2)(s-1)^{t-2} + s(s-1)^{t-1} + 1.$$ □

Corollary. *Setting* $s = t$ *gives*

$$c(s,s,(s-2)^{s-2} \ge 2(s-1)^s + 1, \quad s \ge 3.$$

Theorem 5. *For* $s \ge 2$,

$$b(s,2,[(1/2)(s+1)]) = 0.$$

(That is, if Q *is any subset of* $P(s,2)$ *with* $q(a_i = j) \ge (1/2)(s+1)$, $1 \le i \le 2$, $1 \le j \le s$, *then* Q *contains a complementary set.)*

Proof. The $s \times s$ 0-1 matrix corresponding to Q has at least $(1/2)(s+1)$ 1's in each row and column. Any collection of $s-1$ rows and columns must contain fewer than $(1/2)(s+1)$ rows or fewer than $(1/2)(s+1)$ columns, and hence cannot cover all the 1's. Hence by König's theorem, there are s 1's, no two in the same row or column, and therefore Q contains a complementary set. (An alternative proof can be given using Hall's theorem.) □

Remark. For $1 \le k \le [(1/2)(s-1)]$, let Q_k be the subset of $P(s,2)$ consisting of all pairs ab, cd, where $1 \le a \le k+1$, $1 \le b \le k$, $k+2 \le c \le s$, $1 \le d \le s$. This construction shows that $c(s,2,k) \ge |Q_k| + 1 = (k+1)k + (s-k-1)s + 1$. (For $k = 1$, $s \ne 2$, equality holds by Theorem 2.)

Theorem 6. *For* $s \ge 2$, $k \ge 0$, $t \ge k+2$,

$$c(s,t,s^k) \ge s^t - s^{t-1} - s^k(s-1)^{t-k-1} + s^k + 1.$$

Proof. When $k = 0$ equality holds by Theorem 2. Hence assume $k \geq 1$, and let $B = B_1 \cup B_2$, where

$$B_1 = \{a_1 \cdots a_t \in P(s,t) : a_1 = 1\} - \{a_1 \cdots a_t \in P(s,t) : a_1 = \cdots = a_{t-k} = 1\}$$
$$B_2 = \{a_1 \cdots a_t \in P(s,t) : a_1 = 2, a_2, \cdots, a_{t-k} \neq 1\}.$$

Then B_1 meets every complementary set in $P(s,t)$ except for those met by $\{a_1 \cdots a_t \in P(s,t) : a_1 = \cdots = a_{t-k} = 1\}$, and each of these remaining complementary sets is met by B_2, therefore $Q = P(s,t) - B$ contains no complementary set. When $t \geq k+2$, it is straightforward to check that

$$s^k = q(a_1 = 1) = \min\{q(a_i = j) : 1 \leq i \leq t, 1 \leq j \leq s\},$$

and hence

$$c(s,t,s^k) \geq |Q| + 1 = s^t - s^{t-1} - s^k(s-1)^{t-k-1} + s^k + 1, \quad s \geq 2, k \geq 1, t \geq k+2. \qquad \square$$

3 Remarks, questions, and conjectures

(1) Perhaps Proof 2 of Theorem 1 could be modified so as to give a result concerning $b(s,t,k)$ for $k > 0$.

(2) The construction of Q in Theorem 6 seems very 'efficient'. Perhaps equality holds for all k, and not just for $k = 0$.

(3) The construction which gives $b(s,3,1) > c(s,3,1)$, $s > 2$ (Theorem 3) fails to give $b(s,t,1) > c(s,t,1)$ for any $t > 3$ (proof of Theorem 4). It would be interesting to know if $t = 3$ is any exceptional case, or if $b(s,t,1) > c(s,t,1)$, $s > 2$, for all $t \geq 3$. (If the latter holds, then $t = 2$ is an exceptional case.)

(4) Let $Q(s)$ be the subset of $P(s,s)$ constructed as in the proof of Theorem 4, with $s = t$. Then $Q(s)$ is a 'homogeneous' subset of $P(s,s)$ in the following sense. For each i, j, call the set $\{a_1 \cdots a_s \in P(s,s) : a_i = j\}$ a *hyperplane*. Then for every hyperplane $H(s)$ of $P(s,s)$, either $|Q(s) \cap H(s)| = (1/e + o(1))|H(s)|$ or $|Q(s) \cap H(s)| = (2/e + o(1))|H(s)|$ (as $s \to \infty$). Note that $|Q(s)| = (2/e + o(1))|P(s,s)|$.

Conjecture. For every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that if $s \geq n(\varepsilon)$ and Q is any subset of $P(s,s)$ such that $|Q \cap H| > (1/e + \varepsilon)|H|$ for every hyperplane H of $P(s,s)$, then Q contains a complementary set.

(5) For $s \geq 2$, $t \geq 1$, define $d(s,t)$ to be the smallest integer h with the property that if Q is any subset of $P(s,t)$ such that $h \leq \min\{q(a_i = j) : 1 \leq i \leq t, 1 \leq j \leq s\}$, then Q contains a complementary set. (In terms of partitions and transversals, $d(s,t)$ is the smallest integer h with the property that if $A_1, \ldots, A_t$ are s-cell partitions of the finite set X which separate the points of X, and the smallest of all the cells in all the partitions has at least h elements, then $A_1, \ldots, A_t$ have a common transversal.)

Theorem 5 (and the Remark following) shows that $d(s,2) = [(1/2)(s+1)]$. It would be interesting to find $d(s,t)$, or any upper bound for $d(s,t)$, for $t > 2$.

(6) Perhaps the most interesting of all the open questions is simply this: What is $b(s,3,1)$?

(7) Recently Livingston [3] has shown the following:

$$c(s,t,k) = s^k - s^{t-1} - (s-1)^{t-1} + k + 1, \quad 1 \leq k \leq s-1, s \geq 4, t \geq 3,$$
$$c(s,t,k) = s^t - s^{t-1} - s(s-1)^{t-1} + k + 1, \quad s \leq k \leq s(s-1),$$

and either $s \geq 4$ and $t \geq 4$, or $s = k$ and $t = 3$.

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