A Graph-Theoretic Conjecture Which Implies Szemerédi's Theorem

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Abstract

We briefly review the history of Szemerédi's theorem, and its equivalence to Furstenberg's theorem on multiple recurrence of measure-preserving transformations. We then show that the truth of a certain graph-theoretic conjecture would imply Szemerédi's theorem.

1 Introduction

In 1927, B. L. van der Waerden [10] published a proof of the following theorem, opening an area of research which continues to grow at an explosive rate.

Theorem 1. *For every pair k, r of positive integers there exists an integer $n = n(k, r)$ such that if the set $[1, n] = \{1, 2, \ldots, n\}$ is partitioned in any way into r subsets $A_1, A_2, \ldots, A_r$, then at least one of the subsets A_i must contain a k-term arithmetic progression $a, a+d, a+2d, \ldots, a+(k-1)d$.*

(See [2] for an up-to-date survey of results and current questions of interest in this area, including an extensive list of references.)

In 1975, E. Szemerédi [9] proved the following profound generalization of this result, which had been conjectured in 1936 by Erdős and Turán [3].

Theorem 2. *Let k be an arbitrary positive integer, and let ε be an arbitrary positive real number. Then there exists an integer $n_0 = n_0(k, \varepsilon)$ such that if $n \geq n_0$ and A is any subset of $[1, n]$ which contains more than εn elements, then A contains a k-term arithmetic progression.*

Earlier partial results were obtained by K. F. Roth in 1953 [6], who proved the result for the special case $k = 3$, by E. Szemerédi in 1969 [8], who settled the question for the special case $k = 4$, and by Felix Behrend in 1938 [1], who proved an interesting consequence of the falsity of Theorem 2.

In 1976 H. Furstenberg observed that Theorem 2 is equivalent to the following "multiple recurrence" theorem.

Theorem 3. *Let $(X,\mathscr{B},\mu)$ be a probability measure space. Let T be an invertible, measure-preserving transformation on $(X,\mathscr{B},\mu)$, and let $A \in \mathscr{B}$ be any set of positive measure. Then for every positive integer k there exists a subset B of A with $\mu(B) > 0$ and a positive integer n such that*

$$T^n(B) \cup T^{2n}(B) \cup \cdots \cup T^{(k-1)n}(B) \subset A.$$

Furstenberg found an ergodic theoretical proof of Theorem 3 [5], thus giving a new proof of Szemerédi's Theorem. (A somewhat simplified exposition of this proof is given by Furstenberg, Kaznelso, and Ornstein in [4]).

2 A graph-theoretic conjecture which implies Szemerédi's theorem

Conjecture. Let t,k be arbitrary positive integers. Then there exists a positive integer $N = N(t,k)$ with the following property. Let $n \geq N$, and let the edges of the complete graph on $n+1$ vertices, K_{n+1}, be colred with tn colors, in such a way that no two adjacent edges have been assigned the same color. Then there exists either a bichromatic circuit (a circuit whose edges are colored alternately using only two colors) or a bichromatic path of length k (a path with k edges, which are colored alternately using only two colors).

Proof that the conjecture implies Szemerédi's theorem. Suppose that Szemerédi's theorem is false. Then for some fixed positive integers s and k, and arbitrarily large integers n, there exist sets A_n such that $A_n \subset [1,sn]$, $|A_n| = n+1$, and A_n contains no k-term arithmetic progression.

(We will use these sets A_n to show that $N(2s, 2k-1)$ does not exist, contradicting the conjecture).

It follows that A_n contains no k-term arithmetic progression *modulo* $2sn+1$, that is, A_n does not contain $a_1, a_3, \ldots, a_k$ (not necessarily distinct) such that $a_1 \not\equiv a_2 \pmod{2sn+1}$ and

$$a_2 - a_1 \equiv a_3 - a_2 \equiv \cdots \equiv a_k - a_{k-1} \pmod{2sn+1}.$$

Now let $K(A_n)$ denote the complete graph with vertex set A_n. We color the edges of $K(A_n)$, using the $2sn$ elements of $[1,2sn]$ as colors, in the following way. For x,z in $K(A_n)$, $x \neq z$, we assign to the edge $\{x,z\}$ the unique element y in $[1,2sn]$ such that

$$x + z \equiv 2y \pmod{2sn+1}.$$

Note that this just means that x,y,z form a 3-term arithmetic progression modulo $2sn+1$.

We can now show that there does not exist any bichromatic path of length $2k-1$.

Suppose on the contrary that $a_1 b_1 a_2 b_2 \cdots a_{k-1} b_{k-1} a_k$ are the vertices (in $K(A_n)$) of a bichromatic path of length $2k-1$. Then the edges $\{a_1,b_1\}, \{a_2,b_2\}, \ldots, \{a_{k-1},b_{k-1}\}$ all have say color y, so that

$$2y \equiv a_1 + b_1 \equiv a_2 + b_2 \equiv \cdots \equiv a_{k-1} + b_{k-1} \pmod{2sn+1}.$$

Similarly,

$$b_1 + a_2 \equiv b_2 + a_3 \equiv \cdots \equiv b_{k-1} + a_k \pmod{2sn+1}.$$

In particular, $a_i + b_i \equiv a_{i+1} + b_{i+1} \pmod{2sn+1}$ and $b_{i+1} + a_{i+2} \equiv b_i + a_{i+1} \pmod{2sn+1}$. Adding these two congruences gives

$$a_i + a_{i+2} \equiv 2a_{i+1} \pmod{2sn+1}, \quad 1 \leq i \leq k-2.$$

Hence $a_1, a_2, \ldots, a_k$ form a k-term arithmetic progression modulo $2sn+1$ which is contained in A_n, contradicting one of our assumptions about A_n.

Therefore no bichromatic path of length $2k-1$ exists, for this particular coloring of the edges of $K(A_n)$.

Similarly, no bichromatic circuit can exist, we just go around the circuit (which must have even length) enough times to obtain a k-term arithmetic progression modulo $2sn+1$.

Since these coloring exist for arbitrarily large n, we conclude that $N(2s, 2k-1)$ does not exist. Thus the falsity of Szemerédi's theorem implies the falsity of the conjecture. □

3 Remarks

When $t = 1$, we can take $N(1, k) = 3$. Indeed, if $n \geq 3$ is even then no coloring of K_{n+1} with n colors, where no two adjacent edges have the same color, is possible. If $n \geq 3$ is odd and the edges of K_{n+1} are colored with n colors, where no two adjacent edges have the same color, then for each color, exactly $(n+1)/2$ edges must have the same color. Hence for any two distinct colors, the set of edges having these two colors will be the union of bichromatic circuits.

(These bichromatic circuits can be small. If $n+1 = 2^m$, we can label the vertices of K_{n+1} with the $n+1$ elements of the $(n+1)$-element field, and then use the n non-zero elements of this field as colors by assigning the color $a+b$ to the edge $\{a, b\}$, for each pair $a, b, a \neq b$ of vertices. Under this coloring, every bichromatic circuit is a 4-circuit).

If we use $n+1$ colors for the edges of K_{n+1}, instead of n colors, there need not exist any bichromatic circuits at all. Indeed, let $n+1 = p$, where p is an odd prime. Label the vertices of K_{n+1} with the elemtns of the $(n+1)$-element field, and again assign to the edge $\{a, b\}$ the color $a+b$, for each pair $a, b, a \neq b$, of vertices. Now the zero element of the field is used as a color, and there are no bichromatic circuits. In fact, given any two colors, the set of all edges in these two colors is a path of length $p-1$. (This example is due to Joel Spencer [7]).

At present we are unable to settle the conjecture even for the special case $t = 2$. (Note that the conjecture makes sense even if we take t to be, for example, $3/2$).

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