Probabilistic Prospects of Stackelberg Leader and Follower

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Abstract

In this paper, we consider the family of all $m \times n$ bimatrix games, whose payoff entries are the players' orderings of the outcomes, and count fraction of games whose Stackelberg solution is the leader's hth best outcome and the follower's kth best outcome ($h, k \leq mn$). We conclude that the average leader and follower enjoy symmetric prospects, and that the advantage lies not in the leadership role, but in the relative size of the player's strategy space.

1 Introduction

The Von Stackelberg solution concept for games has found many applications in engineering and economics. It is a staple element of control theory and its extension to dynamic games. In economics, the importance of its original role in understanding monopolistic markets may have dwindled, but our theory of economic design has restaged it in a new and enlarged role.

Typically, this solution concept has been applied in models involving an asymmetry between leader and follower(s) as players, the former enjoying an upper or first hand. Indeed, the folklore of economists may very well associate a higher value with the leadership position than with a followership. On the other hand, it is sometimes not a simple matter to unscramble virtual leaderhip from real leadership and to tell who (if anybody) has an upper hand, as in sharecropping with tenant followers, who may successfully pretend to have different characteristics than their true ones (see Ref. [1]). Quite directly, however, there are many examples where each of two players would prefer being the follower rather than the leader against the other, as seen in the example below.

Example 1.1. Consider the following bimatrix game:

$$\begin{array}{ccc} (3,6) & (4,4) & (9,3) \\ (2,7) & (5,5) & (7,2) \\ (8,1) & (1,8) & (6,9) \end{array}$$

whose entries denote, respectively, the row-player's and column-player's rankings of the outcomes. Under the Von Stackelberge solution concept, the leader (i.e., the player who acts first) attains his 7th best outcome and the follower attains his 2nd best outcome, so that each player would be better off by playing this game as a follower, rather than a leader.

In summary, there is enough reason to question the asymmetry supposedly built into the Von Stackelberg solution concept and often understood to favor the leader. With this motivation, we consider here two-person games whose leaders and followers come from the same population; the former has m possible actions and the latter has n possible actions, each totally ordering the mn combinations as 1st, $\ldots$, mnth best; any leader (resp., follower) is just as likely to play with any given follower (resp., leader) as another. We investigate the probabilities that the hth best outcome for leader and the kth best outcome for follower materialize as Von Stackelberg solution. Among a number of results, Theorem 1 shows a strong *symmetry* between the two roles in obtaining 1st, $\ldots$, mnth best, depending more on m/n than on anything else considered. The results seem to generously justify our skepticism regarding a built-in asymmetry in the value of the leadership and followership roles in the context of the Von Stackelberg solution.

The paper is the beginning of an exploration which we believe should be continued. Our closing remarks seach for directions in this regard, indicating also a natural (Von Stackelberg) type of experiment, giving rise to a family of probability functions (a matter which probabilists may find to be of interest), as they arise in this inquiry.

2 Results

Throughout, m and n will be positive integers, and we will denote

$$\{1,\ldots,m\} = M, \quad \{1,\ldots,n\} = N, \quad \{1,\ldots,mn\} = MN.$$

Define $\mathscr{C}(m,n)$ to be the set of all $m \times n$ matrices whose entries are all distinct members of MN. When m and n are understood, we will feel free to abbreviate $\mathscr{C}(m,n)$ to $\mathscr{C}$. Each pair $\langle A, B\rangle$ of matrices

$$A = (a(i,j)), \quad B = (b(i,j)) \in \mathscr{C}(m,n)$$

is regarded as a *game*. Imagine two players, one with m and the other with n possible actions, and regard A (resp., B) as displaying the 1st (resp., 2nd) player's ranking of the outcomes arising when the 1st player chooses his ith action and the 2nd player chooses his jth action. Given any $B \in \mathscr{C}(m,n)$, we define the *reaction* of B as the function $j_B : M \mapsto N$ with

$$b(i, j_B(i)) = \min_{j\in N} b(i,j), \quad i \in M.$$

The Von Stackelberg *solution* of a game $\langle A, B\rangle$ is the pair

$$(i^*, j^*) = (i^*, j_B(i^*))$$

satisfying

$$a(i^*, j^*) = \min_{i\in M} a(i, j_B(i)).$$

The Von Stackelberg *value* of such a game $\langle A,B\rangle$ is the pair

$$v\langle A,B,\rangle = (r\langle A,B,\rangle, s\langle A,B\rangle) \in MN \times MN,$$

defined through

$$r\langle A,B\rangle = a(i^*,j^*) \text{ and } s\langle A,B\rangle = b(i^*,j^*);$$

and we refer to its 1st (resp., 2nd) coordinate as the Von Stackelberg *value* of the game $\langle A,B\rangle$ to the leader A (resp., follower B).

Given any pair (m,n) and an index set T, for any family

$$\mathscr{G}(m,n) = \{\langle A,B\rangle_t \in \mathscr{C}(m,n) \times \mathscr{C}(m,n) : t \in T\},$$

it is of interest to inquire into the relative frequency of occurrence of a Von Stackelberg value

$$v = (h,k) \in MN \times MN.$$

Such a family (or population) may repeat certain games many times and each with varying frequency. Here, we consider families $\mathscr{G}$ where all games are repeated with the same relative frequency–with no loss of generality, not repeated. Thus, we restric attention to the family $\mathscr{G}$ isomorphic to $\mathscr{C} \times \mathscr{C}$.

The task of our theorem is to evaluate a certain matrix of probabilities and to draw conclusions from this. The matrix in question is given below,

$$\Pi = \begin{matrix} \gamma(n,k) & \alpha(h) & \beta(k) & \\ & \gamma_B(h,k) & \alpha_B(h) & \beta_B(k). \\ & \gamma^A(h,k) & \alpha^A(h) & \beta^A(k) \end{matrix}$$

For any $A,B \in \mathscr{C}$ and any $h,k \in MN$, its entries are defined as follows: $\gamma(h,k)$ (resp., $\gamma_B(h,k), \gamma^A(h,k)$) is the porbabiliity of observing the Von Stackelberg value $v = (h,k)$ (resp., when $B \in \mathbb{C}$ is fixed, when $A \in \mathscr{C}$ is fixed); $\alpha(h)$ (resp., $\alpha_B(H), \alpha^A(h)$) is the probability of observing the leader's Von Stackelberg value $r = h$ (resp., when $B \in \mathscr{C}$ is fixed, when $A \in \mathscr{C}$ is fixed); and $\beta(k)$ (resp., $\beta_B(k), \beta^A(k)$) is the probability of observing the follower's Von Stackelberg value $s = k$ (resp., when $B \in \mathscr{C}$ is fixed, when $A \in \mathscr{C}$ is fixed).

Theorem 1. *The entries of the matrix* Π *are displayed in (1), where*

$$g_i^A(h) = |\{j \in N : a(i,j) > h\}|$$

and $\bar{i}(h)$ *is defined by*

$$a(\bar{i}(h), j) = h, \quad \text{for some } j \in N..$$

$$
\begin{array}{lll}
\alpha(h)\beta(k) & \begin{cases} \binom{mn}{m}^{-1}\binom{mn-h}{m-1}, & \text{if } 1 \le h \le mn-m+1 \\ 0, & \text{if } mn-m+1 < h \le mn \end{cases} & \begin{cases} \binom{mn}{n}^{-1}\binom{mn-k}{n-1}, & \text{if } 1 \le k \le mn+n+1 \\ 0, & \text{if } mn+n+1 < k \le mn \end{cases} \\
\alpha(h)\beta_B(k) & \alpha(h) & \begin{cases} m^{-1}, & \text{if } k = b(i, j_B(i)), \text{ for some } i \in M \\ 0, & \text{otherwise} \end{cases} \\
\alpha^A(h)\beta(k) & m^{-n}\prod_{\substack{i\in M \\ i\neq \tilde{i}(h)}} g_i^A(h) & \beta(k)
\end{array}
\tag{1}
$$

Proof. Take any $h,k \in MN$. First, consider the second row of Π. Take any $B \in \mathscr{C}$, and let

$$b(\tilde{i},\tilde{j}) = k.$$

If $\tilde{j} \neq \underline{j}_B(\tilde{i})$, then there is no leader $A \in \mathscr{C}$ with

$$v\langle A,B\rangle = (h,k),$$

so that

$$\gamma_B(h,k) = 0$$

in this case. Otherwise, consider the set

$$\mathscr{A}_B(h) = \{A \in \mathscr{C} : a(\tilde{i},\tilde{j}) = h \text{ and } a(i,\underline{j}_B(i)) > h \text{ for every } i \in M\},$$

and note that

$$\gamma_B(h,k) = |\mathscr{C}|^{-1}|\mathscr{A}_B(h)|.$$

Now,

$$|\mathscr{C}| = (mn)!;$$

and, counting $\mathscr{A}_B(h)$ is a matter of reckoning in how many ways we can design a matrix $A \in \mathscr{C}$ such that

$$a(\tilde{i},\tilde{j}) = h$$

and the $m-1$ places other than $(\tilde{i},\tilde{j})$ in the graph

$$\Gamma(\underline{j}_B) = \{(i,j) \in M \times N : j = \underline{j}_B(i)\}$$

have entries

$$a(i,j) \in \{h+1,\ldots,mn\}.$$

Thus,

$$|\mathscr{A}_B(h)| = \begin{cases} \binom{mn-h}{m-1}(m-1)!(mn-m)!, & \text{if } 1 \le h \le mn-m+1, \\ 0, & \text{if } mn-m-1 < h \le mn; \end{cases}$$

and so,

$$\gamma_B(h,k) = \begin{cases} m^{-1}\binom{mn}{m}^{-1}\binom{mn-h}{m-1}, & \text{if } 1 \le h \le mn-m+1, \text{ and,} \\ & k = b(i, j_B(i)) \text{ for some } i \in M, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$b(i, \underline{j}_B(i)) = k$$

exactly for m elements $k \in MN$. Hence,

$$\alpha_B(h) = \sum_{k \in MN} \gamma_B(h,k) = \begin{cases} \binom{mn}{m}^{-1}\binom{mn-h}{m-1}, & \text{if } 1 \le h \le mn-m+1 \\ 0, & \text{if } mn-m+1 < h \le mn, \end{cases}$$

so that $\alpha_B(h) = \alpha(h)$. Also note that

$$\sum_{h=1}^{mn-m-1} \binom{mn-h}{m-1} = \binom{mn}{n}.$$

Hence

$$\beta_B(k) = \sum_{h \in MN} \gamma_B(h,k) = \begin{cases} m^{-1}, & \text{if } k = b(i, j_B(i)), \text{ for some } i \in M \\ 0, & \text{otherwise.} \end{cases}$$

Check that

$$\gamma_B(h,k) = \alpha(h)\beta_B(k).$$

Thus, the second row of Π is as given in (1).

Turning next to the third row of Π, take any $A \in \mathscr{C}$, define

$$W^A(h,k) = \{B \in \mathscr{C} : v\langle A, B <\rangle = (h,k)\},$$

and note that

$$\gamma^A(h,k) = |\mathscr{C}|^{-1}|W^A(h,k)|.$$

For each $f \in N^M$, define

$$Z_f(k) = \{B \in \mathscr{C} : \underline{j}_B = f \text{ and } b(i, \underline{j}_B(i)) = k, \text{ for some } i \in M\},$$

and note that

$$Z_f(k) \cap Z_{f'}(k) = \emptyset \quad \text{if } f' \ne f \text{ and } f, f' \in N^M.$$

Denote

$$Z(k) \bigcup_{f \in N^M} Z_f(k).$$

For each $f \in N^M$, also define

$$W_f^A(h,k) = \{B \in Z_f(k) : v\langle A, B\rangle = (h,k)\},$$

and note that $f \neq f'$ implies

$$W_f^Q(k) \cap W_{f'}^Q(k) = \emptyset \quad (f, f' \in N^M).$$

Finally, define

$$\mathscr{F}^A(h) = \{f \in N^M : a(\bar{i}(h), f(\bar{i}(h))) = h \text{ and } a(i, f(i)) \geq h, \text{ for every } i \in M\},$$

and see that

$$|W^Q(h,k)| = \sum_{f \in N^M} |W_f^A(h,k)| = \sum_{f \in \mathscr{F}^A(h)} |W_f^A(h,k)|.$$

There are two important observations to be made at this point. The first is that, for each $f \in \mathscr{F}^A(h)$,

$$|W_f^A(h,k)| = m^{-1}|Z_f(k)|,$$

so that

$$|W^A(h,k)| = m^{-1} \sum_{f \in \mathscr{F}^A(h)} |Z_f(k)|.$$

Secondly,

$$|Z_f(k)| = |Z_{f'}(k)|, \qquad \text{for any } f, f' \in N^M,$$

and so

$$|W^A(h,k)| = m^{-1}|\mathscr{F}^A(h)||N^M|^{-1} \sum_{f \in N^M} |Z_f(k)| = m^{-1}|\mathscr{F}^A(h)||N^M|^{-1}|Z(k)|.$$

Check that

$$|\mathscr{F}^A(h)| = \prod_{\substack{i \in M \\ i \neq \bar{i}(h)}} g_i^A(h), \quad |N^M| = m^n,$$

and compute

$$|Z(k)| = m\binom{mn-k}{n-1} n!(mn-n)!.$$

$Z(k)$ equals the number of ways one can design a matrix $B \in \mathscr{C}$ whose any row has the entry k and $n-1$ other elements from the set $\{k+1, \ldots, mn\}$. Thus,

$$\gamma^A(h,k) = \begin{cases} m^{-1}\left(\prod_{\substack{i \in M \\ i \neq \bar{i}(h)}} g_i^A(h)\right)\binom{mn}{n}^{-1}\binom{mn-k}{n-1}, & \text{if } 1 \leq k \leq mn-n-1, \\ 0, & \text{if } mn-n-1 < k \leq mn. \end{cases}$$

To complete the proof of Theorem 1 for the third row of Π, see that

$$\alpha^A(h) = \sum_{k \in MN} \gamma^A(h,k) = m^{-n} \prod_{\substack{i \in M \\ i \neq \bar{i}(h)}} g_i^A(h).$$

Next, note that

$$\begin{aligned}
&\mathscr{F}^A(h)\cap\mathscr{F}^A(h')=\emptyset, && h,h'\in MN,\\
&f\in N^M\Rightarrow f\in\mathscr{F}^A(h),\& && \text{for some } h\in MN,
\end{aligned}$$

so that

$$\sum_{h\in MN}\prod_{\substack{i\in M\\ i\neq\bar{i}(h)}} g_i^A(h)=\sum_{h\in mn}|\mathscr{F}^A(h)|=|N^M|=m^n.$$

Hence

$$\beta^A(k)=\sum_{h\in MN}\gamma^A(h,k)=\begin{cases}\binom{mn}{n}^{-1}\binom{mn-k}{n-1}, & \text{if } 1\le k\le mn+n+1,\\ 0, & \text{if } mn+n+1<k\le mn,\end{cases}$$

and so $\beta^A(k)=\beta(k)$. It only remains to check that

$$\gamma^A(h,j)=\alpha^A(h)\beta(k),$$

as desired.

Finally, returning to the first row of Π, simply observe that

$$\alpha(h)=|\mathscr{C}|^{-1}\sum_{B\in\mathscr{C}}\alpha_B(h),\quad \beta(k)=|\mathscr{C}|^{-1}\sum_{A\in\mathscr{C}}\beta^A(k);$$

thus

$$\gamma(h,k)=\alpha(h)\beta(k).$$

This completes the proof of Theorem 1 □

To highlight certain aspects of our theorem, a few remarks are in order.

3 Remarks

(i) The fact that

$$\alpha_B(h)=\alpha(h),\quad \text{for any } B\in\mathscr{C},$$

shows that the proportion of leaders $A\in\mathscr{C}$ attaining their hth best outcome at the Von Stackelberg solution is the same regardless of the follower $B\in\mathscr{C}$ with which they are to play. Similarly, regardless of which leader $A\in\mathscr{C}$ they are to play, the proportion of followers $B\in\mathscr{C}$ attaining their kth best outcome is $\beta(k)$, for

$$\beta^A(k)=\beta(k),\quad \text{for every } A\in\mathscr{C}.$$

(ii) An intuitively expected result of the type of optimization involved in our solution concept for games is revealed here in the fact that the functions α and β are decreasing (strictly decreasing on their respective supports). There properties of α and β are shared, of course, by α_B and β^A respectively ($A,B\in\mathscr{C}$); not so, however, for functions α^A and β_B. To see this for α^A, take $m=2mn=3$; consider the matrix

$$A = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & 5 \end{matrix},$$

and compute

$$\alpha_A(3) = 1/9 < 2/9 = \alpha^A(4).$$

Regarding β_B, note that this function is constant on its support, but that its support may have gaps, i.e., need not consist of consecutive integers.

(iii) The function α is discretely convex, in the sense that

$$(1/2)(\alpha(h-1)+\alpha(h+1)) \geq \alpha(h), \quad \text{whenever } 2 \leq \leq mn-1$$

(strictly so, in the sense that this inequality is strict for h in the support of α), and similarly for β.

(iv) It is of interest to note that we obtain the product formulas

$$\gamma(h,k) = \alpha(h)\beta(k), \quad \gamma_B(h,k) = \alpha_B(h)\beta_B(k),$$

$$\gamma^A(h,k) = \alpha^A(h)\beta^A(k),$$

showing that the values r and s are independently distributed, even when A or B is fixed.

(v) A possibly more striking insight offered by our theorem (and one which runs against the prima facie asymmetry build into the Von Stackelberg solution concept for games) is a certain symmetry in the structure of the functions γ; let

$$m' = n, \quad n' = m;$$

imagine

$$\mathscr{C}' = \mathscr{C}(m',n'), \quad \mathscr{G}' = \mathscr{G}(m',n');$$

and finally consider the associated α', β', γ', Now, for any $h,k \in MN$, we see that

$$\alpha(h) = \beta'(h), \quad \beta(k) = \alpha'(k),$$

and so

$$\gamma(h,k) = \gamma'(k,h).$$

Thus, what makes a difference to the probability of a player attaining its lth best outcome ($l = h$ or $l = k$) as Von Stackelberg value is the relative cardinality of the player's action set, rather than whether or not the player enjoys leadership.

In fact, there is a further symmetry to be noted in the prospects of leader and follower. From the convexity of α and β remarked in (iii) above, it follows that there is always a real number

$$l^*(m,n) \in [1,mn],$$

such that, if $m \geq n$, then

$$\alpha(l) \lesseqqgtr \beta(l), \quad \text{for every } l \lesseqqgtr l^*(m,n);$$

and, if $m \geq n$, then the reverse inequalities hold. This reinforces the importance of the relative sizes of m and n as determinant of the players' relative prospects in the context of the Von Stackelberg soltuion.

4 Conclusions

Apart from the exposition in Section 1, the results of Section 2, and the consequences remarked in Section 3, there remain a few matters to be touched upon toward further research which may follow from our present query-answer exercise.

For probabilists, it may be worth noting that a natural sort of experiment has been studied here, where a Von Stackelberg leader and follower, each coming from a certain population, have been confronted with one another in a game, and the value of the Von Stackelberg solution has been examined. Somewhat more involved, at least on the face of it, than the experiment of tossing a coin, this experiment defines a family of probabilities $v \leq (h,k)$ for each pair (m,n) of positive integers. Probabilists may be interested in studying, for example, the limiting distributions as m/n tends to one or another real number. One may also suggest studying the sitributions of values of game-theoretic solutions other than the Von Stackelberg solution, and these with many players of more general types of preferences than the total orderes here assumed over outcomes of joint actions.

Returning to the Von Stackelberg solution, one may ask, as we have done, how the present results are modified when one adopts a pretend-but-perform mechanism (PPM) in the style of Alkan and Sertal (Ref. [1]), i.e., when one allows the follower to declare to have an identity $B \in \mathscr{C}$, possible other than genuine, so long as this player's behavior does not then belie this claim of identity. Counting and computing the numbers desired in these circumstances turns out to be somewhat more involved than in our current tale, and the results do not seem to collapse into any form of proximate simplicity. Nevertheless, we intend to announce some less tidy results in this domain separately.

As to the economic implications of this game-theoretic investigation, the symmetry revealed between leadership and followership in the Von Stackelberg solution indeed runs counter to the attributes of power and advantage which economic folklore associates with the leader. Yet, this symmetry may well exist, owing to the fact that here we have treated any total order as representing a possible player, whereas the preferences of typical economic agents may conform to certain regularities, viz., continuity, convexity, monotonicity. It would be of interest to see what difference it would make in our results to restrict players' preferences to come from such economic domains.

Finally, without attempting to sample the rich literature which has gathered around the Von Stackelberg solution concept and related themes since Von Stackelberg (Refs. [4] and [5]), it may be appropriate to indicate to future researchers two relevant studies worth consulting. We note here the recent duopoly analysis by Ono (Ref. [3]) for its endogenous selection of the leadership-followership roles. See also Moulin (Ref. [2]) for a similar inquiry and an analysis utilizing leadership prospects to identify stable cooperative outcomes.

References

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