Common Transversals for Three Partitions

T. C. Brown

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Abstract

This note contains some questions and a result concerning common transversals for partitions (and in particular for three partitions) of a finite set.

In a famous paper of 1935 [2], Philip Hall gave the first (and still the best!) necessary and sufficient condition for the existence of a system of distinct representatives, or transversal, of a family of sets.

(A set \( T \) is a transversal of the family \( A = (A(1), \ldots, A(s)) \) if there is a bijection \( f \) from \( \{1, \ldots, s\} \) onto \( T \) such that \( f(i) \) is an element of \( A(i) \), \( 1 \leq i \leq s \). Hall’s Theorem, beautiful in its simplicity, states that if \( A = (A(1), \ldots, A(s)) \) is any family of \( s \) sets (not necessarily distinct), then a transversal for the family \( A \) exists if and only if the following condition holds: For each \( k \), \( 1 \leq k \leq s \), the union of any \( k \) of the sets \( A(i) \) contains at least \( k \) elements.)

One of the most singular open questions in transversal theory [4] is the question of whether or not there exists a simple necessary and sufficient condition for the existence of a common transversal for three families.

(If several families of sets are given, say \( A_1, \ldots, A_t \), where \( A = (A(i, 1), \ldots, A(i, s)) \), \( 1 \leq i \leq t \), a set \( T \) is a common transversal of \( A_1, \ldots, A_t \) if \( T \) is a transversal of each \( A_i \), \( 1 \leq i \leq t \). Hall’s theorem immediately gives a simple necessary and sufficient condition for the existence of a common transversal of two families.)

Recently, Judith Q. Longyear [3] discovered an extremely simple sufficient condition for the existence of a common transversal for any number of families. (See [3] for details.)

Among other results, Longyear showed that if \( A_1, A_2 \) are \( s \)-cell partitions of a set \( X \) with the property that distinct elements of \( X \) belong to distinct cells of \( A_1 \), or to distinct cells of \( A_2 \), then \( A_1, A_2 \) have a common transversal if \( |X| > s^2 - 2s + 2 \), and that \( s^2 - 2s + 2 \) is best possible.

This result can be visualized in the following way. Let \( L(s) \) be the \( s \times s \) square of lattice points in the plane defined by \( L(s) = (a_1, a_2) : 0 \leq a_1, a_2 \leq s - 1 \), and let \( X \) be a subset of \( L(s) \). Call the sets \( X \cap \{(a_1, a_2) : a_1 = j\}, 0 \leq j \leq s - 1 \), the columns of \( X \), and the sets \( X \cap \{(a_1, a_2) : a_2 = j\}, 0 \leq j \leq s - 1 \), the rows of \( X \). Then regarding the columns of \( X \) as the cells of a partition \( A_1 \) of \( X \), and regarding the rows of \( X \) as the cells of a partition \( A_2 \) of \( X \), Longyear’s result says that the maximum size of a subset \( X \) of \( L(s) \) such that each row and each column of \( X \) is non-empty and \( X \) does not contain any subset \( T \) meeting each row and each column of \( X \) in exactly one element, is \( |X| = s^2 - 2s + 2 \).

In this note we want to call attention to a number of questions related to this result, and especially to the 3-dimensional case, referred to in the title.
Thus let $M(s)$ be the $s \times s \times s$ cube of lattice points defined by $M(s) = \{(a_1, a_2, a_3) : 0 \leq a_1, a_2, a_3 \leq s-1\}$, and let $X$ be a subset of $M(s)$. The planes of $X$ are the 3s sets $X \cap \{(a_1, a_2, a_3) : a_i = j\}$, $1 \leq i \leq 3$, $0 \leq j \leq s-1$.

What is the maximum size $f(s)$ of a subset $X$ of $M(s)$ such that each plane of $X$ is non-empty and $X$ does not contain any subset $T$ meeting each plane of $X$ in exactly one point?

Taking $X = M(s) \cap \{(x,0,0),(0,y,0),(x,y,z) : x \neq 0, y \neq 0\}$ shows that $2(s-1) + s(s-1)^2 \leq f(s)$.

It is also known ( [3]) that $f(s) \leq s^3 - s^2$. Probably one can show that $f(s) = s^3 - (2 + o(1))s^2$ as $s \to \infty$.

Best of all would be to find the exact value of $f(s)$! (the author is inclined to believe that the construction above is “best possible”, so that $f(s) = 2(s-1) + (s-1)^2s$.)

It is natural to generalize this problem to the $t$-dimensional “cube” $M(s,t) = \{(a_1, \ldots , a_t) : 0 \leq a_i \leq s-1, 1 \leq i \leq t\}$. When $X$ is a subset of $M(s,t)$, the hyperplanes of $X$ are the sets $X \cap \{(a_1, \ldots , a_t) : a_i = j\}$, $1 \leq i \leq t$, $0 \leq j \leq s-1$. What is the maximum size $f(s,t)$ of a subset $X$ of $M(s,t)$ such that each hyperplane of $X$ is non-empty and $X$ does not contain any subset $T$ meeting each hyperplane of $X$ in exactly one point? Is it possible that the computation of $f(s,t)$ for all $s,t$ is an NP-complete problem?

Setting $s = t$, and generalizing the construction above which gives $2(s-1) + (s-1)^2s \leq f(s)$ (see [1] for details) leads to the following conjecture. For every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that if $s \geq n(\varepsilon)$ and $X$ is any subset of $M(s,s)$ with each hyperplane of $X$ containing at least $(1/\varepsilon + \varepsilon)s^{-1}$ points, then $X$ contains a subset $T$ meeting each hyperplane of $X$ in exactly one point (where $e = 2.718 \ldots$).

Other related questions can be found in [1] and [3].

References


