Common Transversals for Three Partitions

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Abstract

This note contains some questions and a result concerning common transversals for partitions (and in particular for three partitions) of a finite set.

In a famous paper of 1935 [2], Philip Hall gave the first (and still the best!) necessary and sufficient condition for the existence of a system of distinct representatives, or transversal, of a family of sets.

(A set T is a *transversal* of the family $A = (A(1), \ldots, A(s))$ if there is a bijection f from $\{1, \ldots, s\}$ onto T such that $f(i)$ is an element of $A(i)$, $1 \leq i \leq s$. Hall's Theorem, beautiful in its simplicity, states that if $A = (A(1), \ldots, A(s))$ is any family of s sets (not necessarily distinct), then a transversal for the family A exists if and only if the following condition holds: For each k, $1 \leq k \leq s$, the union of any k of the sets $A(i)$ contains at least k elements.)

One of the most singular open questions in transversal theory [4] is the question of whether or not there exists a simple necessary and sufficient condition for the existence of a common transversal for three families.

(If several families of sets are given, say $A_1, \ldots, A_t$, where $A = (A(i,1), \ldots, A(i,s))$, $1 \leq i \leq t$, a set T is a *common transversal* of $A_1, \ldots, A_t$ if T is a transversal of each A_i, $1 \leq i \leq t$. Hall's theorem immediately gives a simple necessary and sufficient condition for the existence of a common transversal of two families.)

Recently, Judith Q. Longyear [3] discovered an extremely simple *sufficient* condition for the existence of a common transversal for any number of families. (See [3] for details.)

Among other results, Longyear showed that if A_1, A_2 are s-cell partitions of a set X with the property that distinct elements of X belong to distinct cells of A_1, or to distinct cells of A_2, then A_1, A_2 have a common transversal if $|X| > s^2 - 2s + 2$, and that $s^2 - 2s + 2$ is best possible.

This result can be visualized in the following way. Let $L(s)$ be the $s \times s$ square of lattice points in the plane defined by $L(s) = (a_1, a_2) : 0 \leq a_1, a_2 \leq s - 1\}$, and let X be a subset of $L(s)$. Call the sets $X \cap \{(a_1, a_2) : a_1 = j\}, 0 \leq j \leq s - 1$, the *columns of* X, and the sets $X \cap \{(a_1, a_2) : a_2 = j\}, 0 \leq j \leq s - 1$, the *rows of* X. Then regarding the columns of X as the cells of a partition A_1 of X, and regarding the rows of X as the cells of a partition A_2 of X, Longyear's result says that the maximum size of a subset X of $L(s)$ such that each row and each column of X is non-empty and X does *not* contain any subset T meeting each row and each column of X in exactly one element, is $|X| = s^2 - 2s + 2$.

In this note we want to call attention to a number of questions related to this result, and especially to the 3-dimensional case, referred to in the title.

Thus let $M(s)$ be the $s \times s \times s$ cube of lattice points defined by $M(s) = \{(a_1, a_2, a_3) : 0 \le a_1, a_2, a_3 \le s-1\}$, and let X be a subset of $M(s)$. The *planes of* X are the 3s sets $X \cap \{(a_1, a_2, a_3) : a_i = j\}$, $1 \le i \le 3$, $0 \le j \le s-1$.

What is the maximum size $f(s)$ of a subset X of $M(s)$ such that each plane of X is non-empty and X does not contain any subset T meeting each plane of X in exactly one point?

Taking $X = M(s) \cap \{(x,0,0), (0,y,0), (x,y,z) : x \ne 0, y \ne 0\}$ shows that $2(s-1) + s(s-1)^2 \le f(s)$. It is also known ([3]) that $f(s) \le s^3 - s^2$. Probably one can show that $f(s) = s^3 - (2+o(1))s^2$ as $s \to \infty$. Best of all would be to find the exact value of $f(s)$! (the author is inclined to believe that the construction above is "best possible", so that $f(s) = 2(s-1) + (s-1)^2 s$.)

It is natural to generalize this problem to the t-dimensional "cube" $M(s,t) = \{(a_1, \dots, a_t) : 0 \le a_i \le s-1, 1 \le i \le t\}$. When X is a subset of $M(s,t)$, the *hyperplanes of* X are the sets $X \cap \{(a_1, \dots, a_t) : a_i = j\}$, $1 \le i \le t$, $0 \le j \le s-1$. What is the maximum size $f(s,t)$ of a subset X of $M(s,t)$ such that each hyperplane of X is non-empty and X does not contain any subset T meeting each hyperplane of X in exactly one point? Is it possible that the computation of $f(s,t)$ for all s,t is an NP-complete problem?

Setting $s = t$, and generalizing the construction above which gives $2(s-1) + (s-1)^2 s \le f(s)$ (see [1] for details) leads to the following *conjecture*. For every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that if $s \ge n(\varepsilon)$ and X is any subset of $M(s,s)$ with each hyperplane of X containing at least $(1/e + \varepsilon)s^{-1}$ points, then X contains a subset T meeting each hyperplane of X in exactly one point (where $e = 2.718\dots$).

Other related questions can be found in [1] and [3].

References

[1] T.C. Brown, *Common transversals for partitions of a finite set*, Discrete Math. **51** (1984), 119–124.

[2] Philip Hall, *On representatives of subsets*, J. London Math. Soc. **10** (1935), 26–30.

[3] J.Q. Longyear, *Common transversals in partitioning families*, Discrete Math. **17** (1977), 327–329.

[4] L. Mirsky, *Transversal theory*, Academic Press, New York, 1971.