An Application of Density Ramsey Theory to Transversal Theory

T. C. Brown

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Abstract

We apply results of [1] and [2] to obtain certain sufficient conditions (which involve arbitrarily small "density") for the existence of a "k-transversal" of t s-block partitions of a set X. Along the way, some questions arise of possible independent interest.

1 Introduction and definitions

In this note we prove the two theorems stated below. Section 2 contains some necessary preliminaries, Section 3 contains the proofs, and Section 4 contains some remarks and related questions.

Definition 1. Let X be a set, and let $P_1,\ldots,P_t$ be s-block partitions of X. Then $P_1,\ldots,P_t$ *separate* the points of X if for every pair of elements x,y of X, $x \neq y$, x and y belong to different blocks of at least one of the partitions P_i.

Definition 2. If $P = (P(1),\ldots,P(s))$ is an (ordered) s-block partition of the set X, a *transversal* of P is a set T of s elements of X, one element from each of the blocks $P(i)$, $1 \leq i \leq s$.

Definition 3. Let $P_1,\ldots,P_t$ be s-block partitions of a set X. a *k-transversal* of $P_1,\ldots,P_t$ is an s-element subset T of X with the following two properties. 1) For each i, $1 \leq i \leq t$, T is either a transversal of P_i or is contained in a single block of P_i. 2) For at least k distinct values of i, T is a transversal of P_i.

Definition 4. For $s \geq 2$, $\varepsilon > 0$, $k \geq 1$, $P(s,\varepsilon,k)$ denotes the smallest positive integer (if one exists!) with the following property. If $t \geq P(s,\varepsilon,k)$ and $P_1,\ldots,P_t$ are s-block partitions of a set X which separates the points of X, and $|X| \geq \varepsilon s^t$, then there exists a k-transversal of $P_1,\ldots,P_t$.

Theorem 1. *$P(2,\varepsilon,k)$ and $P(3,\varepsilon,k)$ exist for all $\varepsilon > 0$ and all $k \geq 1$.*

Theorem 2. *Let $s \geq 2$ be fixed. If $P(s,\varepsilon,1)$ exists for all $\varepsilon > 0$ then $P(s,\varepsilon,k)$ exists for all $\varepsilon > 0$ and all $k \geq 1$.*

2 Preliminaries and further definitions

Let $s \geq 2$ and $t \geq 1$ be given, let $A = \{1, \ldots, s\}$ and let A^t be the t-fold cartesian product $A^t = \{a_1 \ldots a_t : a_i \in A, 1 \leq i \leq t\}$.

Let $P_1, \ldots, P_t$ be s-block partitions of a set X which separate the points of X. Order the blocks of each partition P_i in an arbitrary way, say

$$P_i = (P(i,1), \ldots, P(i,s)), \quad 1 \leq i \leq t.$$

Define the mapping g from X into A^t by setting, for each element x of X,

$$g(x) = a_1 \cdots a_t, \quad \text{where } x \in P(1,a_1) \cap \cdots \cap P(t,a_t).$$

Note that g is injective, since $P_1, \ldots, P_t$ separate the points of X.

Definition 5. Let $s \geq 2$, let $A = \{1, \ldots, s\}$, and let k, t be positive integers with $t \geq k$. Consider any $s \times t$ matrix (a_{ij}), $1 \leq i \leq s$, $1 \leq j \leq t$, which has the property that each column of this matrix is either constant (perhaps different constants for different columns) or is some permutation of the elements of A (perhaps different permutations for different columns), and such that *at least* k of the columns are non-constant. Then the s rows of such a matrix, regarded as elements of A^t, from a *k-complementary set* in A^t.

(This definition, and the mapping g above, is essentially due to Judith Q. Longyear [8].)

Remark. Let $P_1, \ldots, P_t$ be s-block partitions of a set X which separate the points of X, and let $Y = g(X)$. It follows from the definition of g that there exists a k-transversal of $P_1, \ldots, P_t$ if and only if Y contains a k-complementary set.

3 Proofs

In view of the preceding Remark, $P(s, \varepsilon, k)$ (Definition 4) is the smallest positive integer (if one exists) with the following property.

If $A = \{1, \ldots, s\}$, $t \geq P(s, \varepsilon, k)$, and Y is any subset of A^t with $|Y| > \varepsilon s^t$, then Y contains a k-complementary set.

To prove Theorem 1, we make use of the following known fact.

Fact. (Corollary to Theorem 1 in [2]). The proof of this fact requires the main result given in [1].) If F is either the 2-element field or the 3-element field, and $k \geq 1$, $\varepsilon > 0$ are given, there exists an integer $n(|F|, \varepsilon, k)$ such that if $t \geq n(|F|, \varepsilon, k)$, V is a t-dimensional vector space over F and Y is any subset of V with $|Y| > \varepsilon |V|$, then Y contains a k-dimensional affine subspace (translate of a k-dimensional vector subspace) of V.

Note that any k-dimensional affine subspace of V contains a 1-dimensional affine subspace in which there are at least k nonconstant coordinates. (Here we are viewing V as F^t.)

Now let $s = 2$ or $s = 3$, let $k \geq 1$, $\varepsilon > 0$ be given, let $A = \{1, \ldots, s\}$, and let $t \geq n(|A|, \varepsilon, k)$. Let Y be any subset of A^t with $Y > \varepsilon s^t = \varepsilon |A^t|$. Then identifying A with the s-element field F, and identifying A^t with the t-dimensional vector space V over F, it follows from the Fact above and the following remark that Y contains a 1-dimensional affine subspace with at least k non-constant coordinates, that is, Y contains a k-complementary set.

This shows that $P(s,\varepsilon,k) \leq n(|A|,\varepsilon,k)$ (for $s=2$ or $s=3$), and completes the proof of Theorem 1.

The proof of Theorem 2 is obtained by a slight modification of the proof of Theorem 1 in [2]. For the sake of completeness we give the modified argument here.

Lemma. *Let $s \geq 2$ and $k \geq 1$ be fixed, and assume that $P(s,\varepsilon,1)$ exist for all $\varepsilon > 0$. Then for each positive integer r, the existence of $P(s,1/(r+1),k)$ implies the existence of $P(s,1/r,k+1)$.*

Proof. Let $A = \{1,\dots,s\}$, let $no = P(s,1/(r+1),k)$, and let e be the number of distinct k-complementary sets in A^{no}. Let $\varepsilon' = 1/(er^2)$, and let $n\iota = P(s,\varepsilon',1)$. We now claim that $P(s,1/r,k+1) \leq no + n\iota$.

To see this, let Y be any subset of $A^{no+n\iota}$ with $|Y| > 1/r \cdot s^{no+n\iota}$. We need to show that Y contains a $(k+1)$-complementary set.

For each $z \in A^{n\iota}$, let W_z denote the set $A^{no} \times \{z\}$. Then

$$A^{no+n\iota} = \cup\{W_z : z \in A^{n\iota}\}.$$

Note that if $|Y \cap W_z| > 1/(r+1) \cdot s^{no}$, then by the definition of no, Y must contain a k-complementary set.

Let u denote the number of elements z in $A^{n\iota}$ such that $|Y \cap W_z| \leq 1/(r+1) \cdot s^{no}$.

Then we have

$$1/r \cdot s^{no+n\iota} < |Y| = \sum |Y \cap W_z| \leq u/(r+1) \cdot s^{no} + (s^{n\iota} - u) \cdot s^{no},$$

hence $u(1 - 1/(r+1)) < s^{n\iota}(1 - 1/r)$, $u < s^{n\iota}(1 - 1/r^2)$, $s^{n\iota}/r^2 < s^{n\iota} - u$.

Therefore there are

$$d = s^{n\iota} - u > s^{n\iota}/r^2$$

elements z in $A^{n\iota}$ such that

$$|Y \cap W_z| > 1/(r+1) \cdot s^{n\iota},$$

and each of these sets $Y \cap W_z$ contains a k-complementary set

$$U_z \times \{z\},$$

where U_z is a k-complementary set contained in A^{no}.

Since there are only e distinct k-complementary sets in A^{no}, at least d/e of the sets $U_z \times \{z\}$ must have the form $U \times \{z\}$ for a fixed k-complementary set U in A^{no}. Let these be

$$U \times \{z_1\}, \dots, U \times \{z_h\},$$

where $h \geq d/e$.

Let $Y' = \{z_1, \ldots, z_h\}$. Then Y' is a subset of $A^{n\iota}$ with

$$|Y'| = h \geq d/e > s^{n\iota}/(er^2) = \varepsilon' \cdot s^{n\iota}.$$

Therefore Y' contains a 1-complementary set. Re-numbering if necessary, let this 1-complementary set be $z_1, \ldots, z_s$.

We now have that Y contains $U \times \{z_1\}, \ldots, U \times \{z_s\}$, where U is a k-complementary set in A^{no} and $\{z_1, \ldots, z_s\}$ is a 1-complementary set in $A^{n\iota}$. If $U = \{w_1, \ldots, w_s\}$, then Y contains the $(k+1)$-complementary set

$$w_1 \times z_1, \ldots, w_s \times z_s.$$

This completes the proof of the Lemma. □

Theorem 2 now follows from the Lemma by induction on k. Indeed, the hypothesis of Theorem 2 is the case $k = 1$. For the induction step, if $P(s, \varepsilon, k)$ exists for all $\varepsilon > 0$ then it exists for all $\varepsilon = 1/(r+1)$, $r \geq 1$, hence by the Lemma $P(s, 1/r, k+1)$ exists for all $r \geq 1$, hence $P(s, \varepsilon, k+1)$ exists for all $\varepsilon > 0$.

4 Remarks and questions

A *combinatorial line* T in A^t (where $A = \{1, \ldots, s\}$) is a 1-complementary set of a very special type. When the t-tuples of T are regarded as the rows of an $s \times t$ matrix, then each column of this matrix is either constant or is a *single fixed* permutation of the elements of A.

The celebrated Hales-Jewett theorem [7] states that if s, r are given there exists a smallest positive integer $HJ(s, r)$ such that if $A = \{1, \ldots, s\}$,$t \geq HJ(s, r)$, and A^t is r-colored (that is, a mapping $c : A^t \mapsto \{1, \ldots, r\}$ is given) then there exists a combinatorial line T in A^t which is monochromatic (that is, the mapping c restricted to T is constant.)

The only known upper bounds for the function $HJ(s, r)$ are extremely large, to say the least. (See [3–6] for elegant proofs, generalizations and applications of the Hales-Jewett theorem, and for further discussion of these bounds. See also the numerous references in the excellent survey article [3].)

Let $f(s, r)$ denote the smallest positive integer such that if $A = \{1, \ldots, s\}$, $t \geq f(s, r)$, and A^t is r-colored then there exists a 1-complementary set in A^t which is monochromatic. Perhaps a "reasonable" (primitive recursive?!) upper bound can found for the function $f(r, s)$.

One could also ask for bounds on the function $f(s, r, k)$, the smallest positive integer such that if $A = \{1, \ldots, s\}$, $t \geq f(s, r, k)$, and A^t is r-colored, there exists a k-complementary set in A^t which is monochromatic.

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