A partition of the non-negative integers, with applications to Ramsey Theory

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Citation data: T.C. Brown, *A partition of the non-negative integers, with applications to Ramsey theory*, Discrete Mathematics and its Applications (Proceedings of the International Conference on Discrete Mathematics and its Applications, Amrita Vishwa Vidyapeetham, Ettimadai Coimbatore, India), Narosa Publishing House, 2006, pp. 79–87.

Abstract

After some general remarks about Ramsey theory, we describe a particular partition of the non-negative integers into infinitely many translates of an infinite set. This partition is used to settle (negatively) the question of the truth of a statement similar in form to the Erdős-Graham canonical version of van der Waerden's theorem on arithmetic progressions. It is also used to give a lower bound for one of the classical van der Waerden functions.

1 Introduction

Ramsey theory is a cohesive sub-discipline of combinatorics. The theme of Ramsey theory is that "complete chaos is impossible". Or, one could say that Ramsey theory is "the study of unavoidable regularities in large structures".

Two of the largest branches of Ramsey theory start with either "Ramsey's Theorem" on the one hand, or "van der Waerden's Theorem on Arithmetic Progressions" on the other. These two branches sometimes overlap, but a great number of results can be placed on one branch or the other.

The simplest form of Ramsey's Theorem says that for every positive integer k there exists a (smallest) positive integer $r(k)$ such that any graph on $r(k)$ vertices contains either a complete subgraph on k vertices (all edges are present) or an independent set of k vertices (no edge is present). A stronger statement, also proved by Ramsey, is that in any graph on infinitely many vertices, there is either an infinite set of vertices A, in which all edges are present, or there is an infinite set of vertices B, in which no edge is present.

The simplest form of van der Waerden's Theorem on Arithmetic Progressions says that for every positive integer k there exists a (smallest) positive integer $w(k)$ such that if $\{1, 2, \ldots, w(k)\}$ is partitioned into two parts, in any way whatsoever, then at least one of the parts contains a k-term arithmetic progression, that is, a subset of the form $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$. (An equivalent statement is the following: If the set of all positive integers is partitioned into two parts, then one part must contain arbitrarily large (finite) arithmetic progressions.)

For example, $w(5) = 178$, and this means:

1. If $\{1,2,3,\ldots,178\}$ is the disjoint union of A and B, then A or B must contain a 5-term arithmetic progression $\{a, a+d, a+2d, a+3d, a+4d\}$.

2. There exists a partition of $\{1,2,3,\ldots,177\}$ into sets A and B such that neither A nor B contains a 5-term arithmetic progression.

The fact that $w(5) = 178$ was shown by direct computation. To establish $w(5) \leq 178$, one has to essentially check all of the 2^{178} partitions into two parts of $\{1,2,3,\ldots,178\}$, so its not surprising that the value of $w(6)$ is unknown. (It is known that $w(6) \geq 696$.) Other known values of $w(k)$ are $w(3) = 9$, $w(4) = 35$, $w(3,3,3) = 27$ (partitions of $\{1,2,\ldots,27\}$ into 3 parts), and $w(3,3,3,3) = 76$ (partitions of $\{1,2,\ldots,76\}$ into 4 parts). There are also a few known values such as $w(4,3,3) = 51$, which means that every partition of $\{1,2,\ldots,51\}$ into 3 parts produces a 4-term arithmetic progression in the first part, or a 3-term arithmetic progression in the 2nd or 3rd parts, and 51 is the smallest positive integer with this property.

In 1999, Ron Graham gave Timothy Gowers a "reward" of \$1000 US dollars for showing that $w(k) < 2^{2^{2^{2^{2^{k+9}}}}}$. This bound, while quite large, is tiny compared to the previous best-known bounds.

The true rate of growth of the function $w(k)$ is one of the holy grails of Ramsey theory, and Ron Graham now offers \$1000 US dollars for a proof (or disproof) that $w(k) \leq 2^{k^2}$. The best lower bound known for $w(k)$ is the following: for every $\varepsilon > 0$, $2^k/k^\varepsilon < w(k)$, for all sufficiently large k.

2 The Description of a Particular Partition

Let S denote the set of all distinct sums of odd powers of 2, including 0 as the empty sum, and let T denote the set of all distinct sums of even powers of 2, including 0 as the empty sum. Then every non-negative integer can be written uniquely in the form $s+t$, where $s \in S$ and $t \in T$. Thus $\{s+T : s \in S\}$ is a partition of $\omega = \{0,1,2,\ldots\}$ into translates of T.

It is more convenient to describe this partition as a *coloring* f of ω. Thus for each $n \in \omega$, we write $n = s+t$, $s \in S$, $t \in T$, and define $f(n) = s$. In other words, if $n = \sum_{i \text{ odd}} 2^i + \sum_{i \text{ even}} 2^i$, then $f(n) = \sum_{i \text{ odd}} 2^i$. For this coloring f, the set of colors is S, and for each $s \in S$, f is constant on the "color class" $s+T$.

3 A van der Waerden-Like Theorem

We need the following definition.

Definition 1. *If* $A = \{a_1 < a_2 < \cdots < a_n\} \subset \omega = \{0,1,2,\ldots\}$, $n > 1$, *the* gap size *of* A *is* $\mathrm{gs}(A) = \max\{a_{j+1} - a_j : 1 \leq j \leq n-1\}$. *If* $|A| = 1$, $\mathrm{gs}(A) = 1$.

Theorem 1. *If ω is finitely colored, there exists a fixed $d \geq 1$ (d depends only on the coloring) and arbitrarily large (finite) monochromatic sets A with $\mathrm{gs}(A) = d$.*

This fact first appeared in [3]. A proof can be found in [10]. Various applications appear in [4,5,9,11].

Theorem 1 is somewhat similar in form to van der Waerden's theorem on arithmetic progressions [13]. However, Theorem 1 differs in a number of ways.

Van der Waerden's theorem does not imply Theorem 1, since the d in the conclusion of Theorem 1 is independent of the size of the monochromatic sets A. Beck [1] showed the existence of a 2-coloring of ω such that if A is any monochromatic arithmetic progression with common difference d, then $|A| < 2\log d$. Hence the presence of large monochromatic arithmetic progressions, which is guaranteed by van der Waerden's theorem, is not enough to imply Theorem 1. Somewhat earlier, Justin [8] found an explicit coloring such that if A is any monochromatic arithmetic progression with common difference d, then $|A| < h(d)$; in his example, the coloring is explicit but the function $h(d)$ is not.

Theorem 1 (which has a simple proof) does not imply van der Waerden's theorem in a simple way.

Theorem 1 does not have a density version corresponding to Szemerédi's theorem [12]. That is, there exists a set $X \subset \omega$ with positive upper density for which there do not exist a fixed $d \geq 1$ and arbitrarily large sets $A = \{a_1 < a_2 < \cdots < a_n\}$ with $\mathrm{gs}(A) = d$. For an example of such a set, see [2].

Finally, recall that the Erdős-Graham canonical version of van der Waerden's theorem ([6]) states that if $g : \omega \to \omega$ is an arbitrary coloring of ω (using finitely many or infinitely many colors) then there exist arbitrarily large arithmetic progressions A such that either g is constant on A, i.e. $|g(A)| = 1$, or g is one-to-one on A, i.e. $|g(A)| = |A|$.

We show that there is no such canonical version of Theorem 1. This is Corollary 1 below.

4 Theorem 1 Does Not Have a Simple Canonical Version

Theorem 2. *For every $A \subset \omega$ (with f as described above, in the "description of a particular coloring"),*

$$\frac{1}{4}\sqrt{|A|/\mathrm{gs}(A)} < |f(A)| < 4\sqrt{|A|\mathrm{gs}(A)}.$$

Corollary 1. *For the coloring f above, there do not exist a fixed d and arbitrarily large sets A with $\mathrm{gs}(A) = d$ on which f is either constant or 1-1.*

Proof of Corollary 1. If $16\mathrm{gs}(A) \leq |A|$, then by Theorem 2, $1 < |f(A)| < |A|$. □

To prove Theorem 2, we need the following definition.

Definition 2. *For $k \geq 0$, an* aligned block *of size 4^k is a set of 4^k consecutive integers whose smallest element is $m4^k$, for some $m \geq 0$.*

Proof of Theorem 2. Note that the first aligned block of size 4^k, namely $[0, 4^k - 1] = [0, 2^{2k} - 1]$, is in 1-1 correspondence with the set of all binary sequences of length $2k$. From this we see (by the definition of f) that for $n \in [0, 2^{2k} - 1]$, there are 2^k possible values of $f(n)$, and each value occurs exactly 2^k times. It is easy to see (using the definition of f) that the same is true for any aligned block $[m4^k, m4^k + 4^k - 1]$.

We express this more simply by saying that *"each aligned block of size 4^k has 2^k colors, each appearing exactly 2^k times"*.

Now we can establish the upper bound in Theorem 2. Let $A = \{a_0 < a_1 < a_2 < \cdots < a_n\} \subset \omega$. Then $a_n \leq a_0 + n\mathrm{gs}(A) = a_0 + (|A|-1)\mathrm{gs}(A)$, or

$$a_n - a_0 < |A|\mathrm{gs}(A).$$

Choose s minimal so that A is contained in the union of two adjacent aligned blocks of size 4^s. (Two blocks are needed in case A contains both $m4^s - 1$ and $m4^s$ for some m.) Then

$$4^{s-1} < a_n - a_0.$$

Since each aligned block of size 4^k has 2^k colors,

$$|f(A)| \leq 2 \cdot 2^s.$$

Putting these three inequalities together gives

$$|f(A)| < 4\sqrt{|A|\mathrm{gs}(A)}.$$

Next, we establish the lower bound for $|f(A)|$, which requires a bit more care. We will use the following Lemma.

Lemma 1. *For each $k \geq 0$, any two aligned blocks of size 4^k (consecutive or not) are either colored identically, or have no color in common.*

Proof of Lemma 1. Consider the aligned blocks $[p4^k, p4^k + 4^k - 1]$ and $[q4^k, q4^k + 4^k - 1]$. By the definition of f (and since 4^k is an even power of 2), $f(p4^k) = f(p)4^k$, so that $f(p4^k) = f(q4^k)$ if and only if $f(p) = f(q)$. Also, for $0 \leq j \leq 4^k - 1$, $f(p4^k + j) = f(p4^k) + f(j)$. This last equality obviously holds if $p = 0$, and for $p > 0$ it holds since then each power of 2 which occurs in j is less that each power of 2 which occurs in $p4^k$. Thus the blocks $[p4^k, p4^k + 4^k - 1]$ and $[q4^k + 4^k - 1]$ are colored identically if $f(p) = f(q)$, and have no color in common if $f(p) \neq f(q)$.

Proceeding with the lower bound in Theorem 2, we note that for $k \geq 1$, the colors of any aligned block of size 4^k have the form $UUVV$, where U and V are blocks of size 4^{k-1}.

Next, we note that any block of size 4^k, aligned or not, contains *at least* 2^k colors. For let A be any block of size 4^k. Let the first element of A lie in the aligned block S of size 4^k, and let T be the aligned block of size 4^k which immediately succeeds S. If S and T are colored identically, then the elements of $f(A)$ are just a cyclic permutation of the elements of $f(S)$, and hence the block A contains exactly 2^k colors. By Lemma 1, the remaining case is when S, T have no color in common. In this case, by the preceding paragraph, $f(S)f(T) = UUVVXXYY$, where no two of U, V, X, Y have a color in common, and U, V, X, Y are of size 4^{k-1}. Then $f(A)$, which has size 4^k, contains either UV or VX or XY, and so has at least $2^{k-1} + 2^{k-1} = 2^k$ colors.

Finally, we note that for $s \geq 1$, $k \geq 1$, every set of 4^s consecutive aligned blocks of size 4^k contains at least 2^s blocks of size 4^k, no two of which have a common color. This follows from the fact that these

4^s blocks have the form $[p4^k, p4^k+4^k-1]$, $t \le p \le t+4^s-1$, for some t. The block $f([t,t+4^s-1])$ has at least 2^s colors, by the preceding paragraph. If $f(p) \ne f(q)$, where $t \le p < q \le t+4^s-1$, then $f(p4^k) \ne f(q4^k)$, so by Lemma 1 the two blocks $[p4^k, p4^k+4^k-1]$ and $[q4^k, q4^k+4^k-1]$ have no color in common.

Now let $A \subset \omega$ be given. Choose k so that $4^{k-1} \le \mathrm{gs}(A) < 4^k$. Choose t minimal so that A is contained in the union of t consecutive aligned blocks of size 4^k. Then A meets each of these blocks (by the choice of k), and

$$|A| \le t4^k.$$

Choose s so that $4^s \le t < 4^{s+1}$. Then among the t consecutive aligned blocks of size 4^k are at least 2^s blocks of size 4^k, no two of which have a color in common. Since each of the t blocks meets A, we have

$$2^s \le |f(A)|$$

Thus $|A| \le t4^k < 4 \cdot 4^s \cdot 4 \cdot 4^{k-1} \le 4|f(A)|^2 \cdot 4 \cdot \mathrm{gs}(A)$, so $\frac{1}{4}\sqrt{|A|/\mathrm{gs}(A)} < |f(A)|$. □

□

5 An Easy Lower Bound for a Classical van der Waerden Function

Definition 3. *For $m \ge 1$, let $w(3;m)$ denote the smallest positive integer such that every m-coloring of $[1, w(3;m)]$ produces a monochromatic 3-term arithmetic progression.*

Theorem 3. *For all $m \ge 1$, $w(3;m) \ge \frac{1}{4}m^2$.*

Proof of Theorem 3. The coloring f shows (since the set T contains no 3-term arithmetic progression) that for $k \ge 1$, $w(3;2^k) > 2^{2k}$. For a general m, choose k so that $2^k \le m < 2^{k+1}$. Then $w(3;m) \ge w(3;2^k) > 2^{2k} > \frac{1}{4}m^2$. □

6 Remarks

1. The lower bound in Theorem 3 is not the best possible. Indeed, in the standard reference Ramsey Theory [7], the authors show with more elaborate techniques that for some positive constant c, $w(3;m) > m^{c\log m}$.

2. Corollary 1 shows that a constant/1-1 canonical version of Theorem 1 is not true. We also know by the Bergelson/Hindman/McCutcheon example that a density version of Theorem 1 is not true. The following three simple examples, most involving only 3-element sets, illustrate various combinations of the truth or falsity of the "constant/1-1 version" and the "density version".

    (a) The simplest non-trivial case of van der Waerden's theorem says that every finite coloring of the positive integers produces a monochromatic 3-term arithmetic progression. The constant/1-1 version of the result holds by the Erdős-Graham theorem, and the density version holds by Szemerédi's theorem.

(b) Schur's theorem says that if the positive integers are finitely colored, then there is a monochromatic solution of $x+y=z$. The density version does not hold by taking all the odd integers. The constant/1-1 version does not hold by coloring each x with the highest power of 2 dividing x.

(c) Kevin O'Bryant showed me this example: If the positive integers are finitely colored, then there is a monochromatic 3-term geometric progression (a set of the form $\{a, ad, ad^2\}$). To get the constant/1-1 version, let a coloring g of the positive integers be given. Define a new coloring h by setting $h(x) = g(2^x)$. Then, by the Erdős-Graham theorem, there is a set $\{a, a+d, a+2d\}$ on which the coloring h is either constant or 1-1, so the coloring g is either constant or 1-1 on the set $\{2^a, 2^a 2^d, 2^a (2^d)^2\}$. The density version does not hold, since the set of square-free numbers has positive density.

(d) It seems natural to ask for a collection P of 3-element sets (if such a collection exists!) for which:

   i. Every set of positive integers with positive upper density contains an element of P.

   ii. It's not the case that for every coloring of the positive integers, there is an element of P on which the coloring is either constant or 1-1. Allen R. Freedman communicated the following example to me, involving infinite sets. Here instead of considering those collections of triples $\{x, y, z\}$ for which $x+z=2y$, or $x+y=z$, or $xz=y^2$, one considers the collection P of all subsets of ω which have positive density. Then trivially every set of positive density contains an element of P. However, any coloring which is constant on each interval $[2^{n-1}, 2^n]$, with different colors for different n, shows that the constant/1-1 version does not hold.

3. We have used a particular partition of ω into infinitely many translates of an infinite set. Perhaps it's possible to describe *all* partitions of ω into infinitely many translates of an infinite set.

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