Common Transversals

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Abstract

A 2-coloring of the non-negative integers and a function h are given such that if P is any monochromatic arithmetic progression with first term a and common difference d then $|P| \le h(a)$ and $|P| \le h(d)$. In contrast to this the following result is noted. For any k, f there is $n = n(k, f)$ such that whenever n is k-colored there is a monochromatic subset A of n with $|A| > f(d)$, where d is the maximum of the differences between consecutive elements of A.

1 Introduction

Paris and Harrington [5] have shown that the following simple modification of the finite version of Ramsey's theorem, which can be deduced from the infinite version by a diagonalization argument, is not provable in Peano's first order axioms, even in the case where f is the identity function: Let $r, k \in \omega$, $f \in \omega^\omega$ be given. Then there is $n = n(k, r, f)$ such that whenver $[n]^r$ is k-colored there is a subset A of n with $[A]^r$ monochromatic and $|A| > f(a_0)$, where a_0 is the smallest element of A.

It seems natural to ask whether van der Waerden's theorem on arithmetic progressions can be modified in the sam way. That is, given $k \in \omega$, $f \in \omega^\omega$, must there exist $n = n(k, f)$ such that whenever n is k-colored there is a monochromatic arithmetic progression $P = \{a, a+d, a+2d, \dots\}$ contained in n such that $|P| > f(a)$ or $|P| > f(d)$?

Fact 1 below shows that this question has a negative answer. In contrast to this we quote a result (Fact 3) which shows that if "arithmetic progression with common difference d" is replaced by "set with maximum difference between consecutive elements equal to d" then the corresponding question has an affirmative answer (Furthermore, this result has a simple inductive proof.)

2 The nagative result concerning arithmetic progressions

Fact 1. *There is a 2-coloring of ω and a function h such that if $P = \{a, a+d, \dots\}$ is any monochromatic arithmetic progression then $|P| \le h(a)$ and $|P| \le h(d)$.*

In what follows, the notation $\bar{z}$ will be used for the fractional part of z, i.e., $\bar{z} = z - \lfloor z \rfloor$ for real numbers z. Intervals in ω of the form $[2^k, 2^{k+1})$ (as well as the set $\{0\}$) will be referred to as "blocks." Three obvious lemmas will be used.

Lemma. *If n multiples of d' are contained in $[2^k, 2^{k+1})$ then at least $2n-1$ multiples of d' are contained in $[2^{k+1}, 2^{k+2})$.*

Lemma. *Given integers $a \geq 0$ and $d \geq 2$, let $d' = 2^{p+1}d$ if $a \in [2^p, 2^{p+1})$, $d' = d$ if $a = 0$. Then for each $m \in \omega$, both md' and $a + md'$ belong to the same block.*

Lemma. *Let x, y be real with y irrational. Let $n, s, t \in \omega$ with $n \geq 2$, $s \geq 2n-1$, $t \geq 2s-1$. Let $S_1 = \{\overline{x+my} : m \in [0,n)\}$, $S_2 = \{\overline{x+my} : m \in [n, n+s)\}$, $S_3 = \{\overline{x+my} : m \in [n+s, n+s+t)\}$. Then it is impossible to have simultaneously $S_1 \subset [0,1/2)$, $S_2 \subset [1/2,1)$, $S_3 \subset [0,1/2)$. (The same conclusion hods if $[0,1/2)$ and $[1/2,1)$ are interchanged.)*

Now we define a “preliminary 2-coloring of ω. Let $\alpha > 0$ be fixed and irrational, and define $c_1 : \omega \mapsto \{0,1\}$ by $c_1(n) = 0$ if $\overline{n\alpha} \in [0,1/2)$, $c_1(n) = 1$ if $\overline{n\alpha} \in [1/2,1)$. Suppose that $P = \{a, a+d, \dots\}$ is a monochromatic (with respect to c_1) arithmetic progression with common difference d (and first term a). It follows immediately from the density in $[0,1]$ of the set $\{\overline{md\alpha} : m \in \omega\}$ (and from the density in $[0,1]$ of any translate of this set by $\overline{a\alpha}$ (modulo 1)) that there is $f(d)$ such that $|P| \leq f(d)$, independent of a.

We are now ready to define the 2-coloring $c : \omega \mapsto \{0,1\}$ whose existence is asserted in Fact 1. The colroing c is obtained by starting out with the coloring c_1 and then “reversing" this coloring on alternate blocks. That is, let $c_2(n) = 1 - c_1(n)$ and define, for $n \in [2^k, 2^{k+1})$, $c(n) = c_1(n)$ if k is odd, $c(n) = c_2(n)$ is k is even. (Set $c(0) = c_1(0) = 0$.)

Let $P = \{a, a+d, \dots\}$ be any arithmetic progression which is monochromatic with respect to the coloring c. We show first that $|P|$ is bounded by a function of d, and then that $|P|$ is bounded by a function of a.

To show that $|P|$ is bounded by a function of d, let $f(d)$ be as above and choose k so that $2^k \leq df(d) < 2^{k+1}$. If P intersects both $[0, 2^{k+1})$ and $[2^{k+2}, \infty)$, the block $[2^{k+1}, 2^{k+2})$ will contain more than $f(d)$ consecutive terms of P. (This follows from Lemma 2 and $f(d) \geq 2$.) Since c agrees on $[2^{k+1}, 2^{k+2})$ with either c_1 or c_2, this contradicts the definition of $f(d)$. Hence either $P \subset [2^{k+1}, \infty)$ or $P \subset [0, 2^{k+2})$. In the first case one gets $|P| \leq 2f(d)$, and in the second case $|P| \leq 2^{k+2}/d + 1 \leq 4f(d) + 1$.

Next, to show that $|P|$ is bounded by a function of a, we shall derive a contradiction by assuming that $|P| \geq 32 \cdot 2^{p+1} + 1$ if $a \in [2^p, 2^{p+1})$ and $|P| \geq 33$ if $a = 0$.

Let d' be as in Lemma 2, and consider the progressions $P' = \{a+d', a+2d', \dots, a+32d'\}$, $P'' = \{d', 2d', \dots, 32d'\}$. Choose k so that $2^k \leq 4d' < 2^{k+1}$, and let $A = [2 \cdot 2^{k+1})$, $B = [2^{k+1}, 2^{k+2})$, $C = [2^{k+2}, 2^{k+3})$. We say that $4d'$ belongs to “block A." Now $2d' \in [2^{k-1}, 2^k)$, hence (applying Lemma 2 if necessary) it is clear that block A contains n consecutive elements of P'', where $n \geq 2$. Since by Lemma 2 the elements of P' are distributed among the various blocks in exactly the same way as are the corresponding elements of P'', we obtain that the blocks A, B, C contain respectively n, s, t elements of P', where $n \geq 2$, $s \geq 2n-1$, $t \geq 2s-1$. (Note that the progression P' extends beyond block C since $2^{k+3} \leq 32d'$.)

Now let $a + ud'$ be the first element of $P' \cap (A \cup B \cup C)$, and let $x = (a+ud')\alpha$, $y = d'\alpha$. Since P' is monochromatic with respect to c, the first n terms, next s terms, next t terms, of the sequence $(\overline{x}, \overline{x+y}, \overline{x+2y}, \dots)$ must be contained respectively in the intervals $[0, \frac{1}{2})$, $[\frac{1}{2}, 1)$, $[0, \frac{1}{2})$ (or in $[\frac{1}{2}, 1)$, $[0, \frac{1}{2})$, $[\frac{1}{2}, 1)$), finally contradicting Lemma 2.

This completes the proof of Fact 1; for the function h we can take $h(x) = \max\{4f(x)+1, 64x\}$.

3 The positive result concerning sets with given gap size

Although the results noted here are not new, Fact 3 provides an interesting contrast to the negative result above. The proofs are omitted. Fact 3, the finite version of Fact 2, can be proved by a simple induction on the number of colors.

Let $A = \{a_1, \ldots, a_m\}$ be a finite subset of ω, with $a_1 < \cdots < a_m$. Define $gs(A)$, the "gap size" of A, by $gs(A) = \max\{a_{j+1} - a_j : 1 \leq j < m\}$ if $|A| > 1$ and $gs(A) = 1$ if $|A| = 1$.

Fact 2. *Let $k \in \omega$ and a k-coloring of ω be given. Then there exist $d \in \omega$ and arbitrarily large (finite) monochromatic sets A with $gs(A) = d$.*

Fact 3. *Let $k \in \omega$, $f \in \omega^\omega$ be given. Then there is n such that if n is k-colored there is a monochromatic subset A of n with $|A| > f(gs(a))$.*

We remark that if $n(k, f)$ is the smallest such n, then $n(1, f) \leq 1 + f(1)$ and $n(k, f) \leq 1 + kf(n(k-1, f))$. (Letting e denote the identity function, this gives $n(k, e) \leq k!(1 + 1/1! + \cdots + 1/k!)$, while in fact $n(k, e) = k^2 + 1$; hence the above bound is far from best possible.)

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Note added in proof. The author completely overlooked Justin's very different construction ([3]– long before the Paris and Harrington result) of a 2-coloring of ω such that any arithmetic progression P with common difference d has $|P|$ bounded by a function of d, namely, for each $n \geq 1$, let $n! = 2^t q$, q odd, and define $c(n) = 0$ if t is even, $c(n) = 1$ if t is odd.

References

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