On Homogeneous Cubes

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Citation data: T.C. Brown, *On homogeneous cubes*, Bogazici University J. **6** (1978), 13–16.

Abstract

There exists an integer $n \leq 10^{11}$ with the following property. Let the ordinary $n \times n$ square in the plane be divided into 1×1 “cells", and let each of these cells be colored either red or blue in any way whatsoever. Then there is a color, say red, an integer d, and a $10^4 \times 10^4$ sub-square S of the original square such that every $d \times d$ sub-square of S contains a red cell.

A generalized version of this fact is proved.

1 Notation and Definitions

If Y is any set, a *k-coloring* of Y is an asignment of one of k different “colors" to each of the elements of Y, That is, a k-coloring of Y is a function c from Y to a set of k colors, say $c_1, c_2, \dots, c_k$. A subset B of Y is called *monochromatic* if all the elements of B have been assigned the same color, that is, if for some i, $c(x) = c_i$ for all x in B.

The notations $\omega = \{0, 1, 2, \dots\}$ and $p = \{0, 1, \dots, p-1\}$ will be used. As usual, for any set X, the symbol Y^X denotes the set of all functions from X to Y. If X has only a finite number t of elements, then ω^X is denoted by ω^t.

An N-cube in ω^X is any set constructed in the following way. For each element x in X, choose an integer $a(x)$. Then the set $\{y \text{ in } \omega^X : a(x) \leq y(x) \leq a(x) + N\}$ is an N-cube. Note that an N-cube in ω^3 is an ordinary $N \times N \times N$ cube.

2 Statement of Results

Definition. Let a k-coloring of ω^X be given. An N-cube X in ω^X is called *d-homogeneous* is every $(d-1)$-cube contained in C contains an element which has been assigned some one fixed color.

Theorem 1. *Given a positive integer k, a set X, and any function f from ω to ω, there exists an integer $n = n(k, X, f)$ such that if n^X is k-colored in any way whatsoever then n^X contains a d-homogeneous $f(d)$-cube, for some integer d. Furthermore, if n is chosen as small as possible, then $n(1, X, f) = f(1)$ and $n(k, X, f) \leq f(1 + n(k-1, X, f))$ for all $k \geq 2$. (It may be noted that these bounds are independent of X.)*

Theorem 2. *Given a positive integer k and a set X, let ω^X be k-colored in any way whatsoever. Then there exist d and d-homogeneous N-cubes for arbitrarily large N.*

3 Proofs

Proof of Theorem 1. Clearly $n(1,X,f)$ exists and equals $f(1)$. Suppose $n(k-1,X,f)$ exists and equals m. Suppose now that n^X has been k-colored in such a way that every d-homogeneous N-cube in n^X has $N < f(d)$. Let D be any M-cube in n^X which uses only $k-1$ colors. Then by assumption, $M \leq n(k-1,X,f)-1 = m-1$. Hence every m-cube in n^X uses every color, hence n^X itself is an $(m+1)$-homogenous n-cube. By assumption, this implies $n < f(m+1)$, so (having chosen n as large as possible) $n(k,X,f) = n+1 \leq f(m+1) = f(1+n(k-1,X,f))$. □

Proof of Theorem 2. Suppose the theorem is false. Then there are an integer k, a set X, and a particular k-coloring of ω^X such that for every d there is an $f(d)$ such that whenever C is a d-homogeneous N-cube then $N < f(d)$. That is, there is no d-homogeneous $f(d)$ cube, and this contradicts Theorem 1. □

References

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[2] ______, *On van der Waerden's theorem and a theorem of Paris and Harrington*, J. Combin. Theory Ser. A **30** (1981), 108–111.