# Variations on van der Waerden's and Ramsey's Theorems

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## 1 Van der Waerden's Theorem

The particular variation of van der Waerden's Theorem to be presented here has been discovered independently by any number of people, but a proof of its equivalence to van der Waerden's Theorem has, to the author's knowledge, never appeared in print. (The variation is stated in [6], and a recent application will appear in [5].)

**Theorem V (Van der Waerden [12]).** *For all positive integers $k$ and $l$ there exists $n = n(k,l)$ such that if any set of $n$ consecutive integers is partitioned into $k$ subsets, at least one of these subsets contains an arithmetic progression of length $l$.*

**Theorem V' (Variation).** *For all positive integers $m$ and $l$ there exists $p = p(m,l)$ such that if $a_1 < a_2 < \cdots < a_p$ are positive integers such that $a_{j+1} - a_j \leq m$, $1 \leq j \leq p-1$, then $\{a_1, \ldots, a_p\}$ contains an arithmetic progression of length $l$.*

Most published proofs of van der Waerden's theorem are carried out by induction on $k$ and $l$. It wuld be of interest to find a direct inductive proof of Theorem V'. In this note, however, we shall show that Theorem V implies Theorem V' and conversely.

To this end, it is convenient to let $V(k,l)$ denote the statement that if the set $N$ of all positive integers is partitioned into $k$ subsets, then at least one of these subsets contains an arithmetic progression of length $l$. Also, let $V'(m,l)$ denote the statement that if $a_1 < a_2 < \ldots$ are a sequence of positive integers such that $a_{j+1} - a_j \leq m$, $j = 1, 2, \ldots$, then the set $\{a_1, a_2, \ldots\}$, contains an arithmetic progression of length $l$.

Clearly Theorem V implies statement $V(k,l)$ for every $k$ and $l$, and Theorem V' implies statement $V'(m,l)$ for every $m$ and $l$. We show now that for every $k$ and $l$, $V(k,l)$ implies the existence of $n(k,l)$. Indeed, suppose that the integer $n = n(k,l)$ does not exist. Then for every $n$ we have a sequence of length $N$ on $k$ symbols which represents a partition of $\{1, 2, \ldots, n\}$ into $k$ subsets such that no subset contains a progression of length $l$.

Let $a_1$ be one of the $k$ symbols which is the 1st symbol of infinitely many of these sequences. Let $a_2$ be a symbol which is the 2nd symbol of infinitely many sequences beginning with $a_1$. Let $a_3$ be the 3rd symbol of infinitely many sequences starting with $a_1 a_2$. In this way we construct an infinite sequence $a_1 a_3 \cdots$ on $k$ symbols which represents a partition of $N$ into $k$ subsets, none of which contains an arithmetic progression of length $l$, contradicting $V(k,l)$. (A similar argument shows that $V'(m,l)$ implies the existence of $p(m,l)$. However, this fact will not be used here.)

We are now ready to show that Theorem V' implies Theorem V. We fix $l$ and demonstrate the existence of $n(k,l)$ by induction on $k$. The case $k=1$ is trivial, so we assume that $n(k,l)$ exists and use this to establish $V(k+1,l)$, which, as noted above, implies the existence of $n(k+1,l)$, thus completing the induction.

Hence, let $N$ be partitioned into $k+1$ subsets and consider the $(k+1)st$ subset $\{a_1,a_2,\dots\}$. We may as well assume it is infinite. If for some $m$, $a_{j+1}-a_j \le m$, $j=1,2,\dots$, then by $V'(m,l)$ (which holds since we are assuming Theorem V') the $(k+1)$st subset contains an arithmetic progression of length $l$. If no such $m$ exists, then the given partition of $N$ induces partitions of arbitrarily large sets of consecutive positive integers into $k$ subsets only. But then by the existence of $n(k,l)$, at least one of these subsets contains an arithmetic progression of length $l$. Thus at least one of the $k+1$ subsets into which $n$ was partitioned contains an arithmetic progression of length $l$. This establishes $V(k+1,l)$, and as previously remarked, completes the induction.

Conversely, we now show that Theorem V implies Theorem V'. Let $m,l$ be given, and let $p=n(m,l)-(m-1)$. Let $A_0=\{a_1,a_2,\dots,a_p\}$, where $a_1<a_2<\dots<a_p$ and $a_{j+1}-a_j\le m$, $1\le j\le p-1$. We show $A_0$ contains an arithmetic progression of length $l$. Define

$$\begin{aligned} A_1 &= \{a_1+1,a_2+1,\dots,a_p+1\}\setminus A_0,\\ A_2 &= \{a_1+2,a_2+2,\dots,a_p+2\}\setminus (A_0\cup A_1),\cdots,\\ A_{m-1} &= \{a_1+m-1,a_2+m-1,\dots,a_p+m-1\}\setminus (A_0\cup A_1\cup\dots\cup A_{m-2}). \end{aligned}$$

Then $\{1,2,\dots,a_p+m-1\}$ is partitioned into the $m$ sets $A_0,A_1,\dots,A_{m-1}$. Since $a_p+m-1\ge p+m-1=n(m,l)$, at least one of these sets, say $A_i$, contains an arithmetic progression of length $l$. Since

$$A_i \subset \{a_1+i,a_2+i,\dots,a_p+i\},$$

the set $A_0=\{a_1,a_2,\dots,a_p\}$ also contains such a progression.

This completes the proof of the equivalence of Theorems V and V'.

## 2 Ramsey's Theorem

Let $G$ be a graph with an infinite number of vertices such that at least two of every three vertices of $G$ are joined by an edge of $G$. G. Szekeres ( [10], [11]) showed that such a graph $G$ must contain an infinite complete subgraph. (An infinite complete subgraph of $G$ is an infinite set of vertices of $G$, *every* two of which are joined by an edge of $G$.)

P. Turán ( [10], [11]) generalized this result by showing that for every fixed positive integer $d$, if $G$ is a graph with an infinite number of vertices such that at least two of every $d$ vertices of $G$ are joined by an edge of $G$, then $G$ contains an infinite complete subgraph.

This result of Turán can itself be further strengthened to obtain the following result, which is designated Theorem R', since it is a “variation" on Ramsey's Theorem.

**Theorem R' (Ramsey [9]).** *Let G be a graph with an infinite number of vertices such that at least two of every* infinite *set of vertices of G are joined by an edge of G. Then G contains an infinite complete subgraph.*

Ramsey's Theorem is the following.

**Theorem R.** *If G is a graph with an infinite number of vertices, then either G contains an infinite complete subgraph or there is an infinite set of vertices of G no two of which are joined by an edge of G.*

Thus Theorem R' is in fact Ramsey's Theorem itself.

*Added in proof:* Another paper has recently appeared in which Theorem V' is mentioned. It is: John R. Rabung, On applications of van der Waerden's theorem, *Math. Magazine*, 48 (1975) 142–148. Rabung's paper contains, amongst other interesting things, an argument essentially identical with the proof above that Theorem V implies Theorem V'.

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