A Proof of Sperner's Lemma via Hall's Theorem

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Sperner's Lemma. Let S be a Sperner set of subsets of $\{1,2,\ldots,n\}$. (That is, for $A,B \in S$, if $A \neq B$ then $A \not\subset B$ and $B \not\subset A$.) Then $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Lemma 1. *Let $A_1,\ldots,A_m$ be distinct k-element subsets of $\{1,2,\ldots,n\}$, where $k \geq \lfloor n/2 \rfloor + 1$. Let $\mathscr{B}_i = \{B \subset A_i : |B| = k-1\}$ $(1 \leq i \leq m)$. Then $\{\mathscr{B}_1,\ldots,\mathscr{B}_m\}$ has a system of distinct representatives.*

Proof of lemma. We use Hall's Theorem, and to simplify notation we show only that $|\mathscr{B}_1 \cup \cdots \cup \mathscr{B}_m| \geq m$. Let $B \in \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_m$, and let

$$e = \{i : B \in \mathscr{B}_i\} = \{i : B \subset A_i\}.$$

Since the A_i are distinct k-element sets and $|B| = k-1$, we have $n \geq k-1+|e|$, and this with $k \geq \lfloor n/2 \rfloor + 1$ gives $|e| \leq k$. Then since $|\mathscr{B}_i| = k$ $(1 \leq i \leq m)$, we obtain $|\mathscr{B}_1 \cup \cdots \cup \mathscr{B}_m| \geq km/k = m$. □

Proof of Sperner's Lemma. Let $k = \max\{|A| : A \in S\}$. If $k \geq \lfloor n/2 \rfloor + 1$, let

$$\{A_1,\ldots,A_m\} = \{A \in S : |A| = k\}.$$

By the lemma, there are distinct sets $\phi(A_i)$ $(1 \leq i \leq m)$, such that $\phi(A_i) \subset A_i$ and $|\phi(A_i)| = k-1$. Thus in S we may replace each A_i by $\phi(A_i)$ $(1 \leq i \leq m)$, to obtain a new Sperner set S' such that $|S'| = |S|$ and $A \in S' \Rightarrow |A| \leq k-1$. Repeating this process as many times as required, we obtain a Sperner set S^* such that $|S^*| = |S|$ and $A \in S^* \Rightarrow |A| \leq \lfloor n/2 \rfloor$.

Now let $T = \{\{1,2,\ldots,n\} \setminus A : A \in S^*\}$. Then T is a Sperner set, $|T| = |S|$, and

$$A \in T \Rightarrow |A| \geq \lfloor n/2 \rfloor.$$

Thus, applying the above process to T, we obtain finally a Sperner set T^* such that $|T^*| = |S|$ and $A \in T^* \Rightarrow |A| = \lfloor n/2 \rfloor$. Thus $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$. □

References

[1] Philip Hall, *On representatives of subsets*, J. London Math. Soc. **10** (1935), 26–30.

[2] D. Lubell, *A short proof of Sperner's lemma*, J. Combinatorial Theory **1** (1966), 299.

[3] E. Sperner, *Ein satz über untermengen einer endlichen menge*, Math. Z. **27** (1928), 544–548.