A Proof of Sperner’s Lemma via Hall’s Theorem

T. C. Brown


Sperner’s Lemma. Let $S$ be a Sperner set of subsets of $\{1, 2, \ldots, n\}$. (That is, for $A, B \in S$, if $A \neq B$ then $A \nsubseteq B$ and $B \nsubseteq A$.) Then $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Lemma 1. Let $A_1, \ldots, A_m$ be distinct $k$-element subsets of $\{1, 2, \ldots, n\}$, where $k \geq \lfloor n/2 \rfloor + 1$. Let $B_i = \{B \subseteq A_i : |B| = k-1\}$ ($1 \leq i \leq m$). Then $\{B_1, \ldots, B_m\}$ has a system of distinct representatives.

Proof of lemma. We use Hall’s Theorem, and to simplify notation we show only that $\bigcup_{i=1}^m B_i$ has a system of distinct representatives. Let $B_2 \supseteq \cdots \supseteq B_m$, and let $e = \{i : B \in B_i \} = \{i : B \subseteq A_i\}$.

Since the $A_i$ are distinct $k$-element sets and $|B| = k-1$, we have $n \geq k-1 + |e|$, and this with $k \geq \lfloor n/2 \rfloor + 1$ gives $|e| \leq k$. Then since $|B_i| = k$ ($1 \leq i \leq m$), we obtain $|B_1 \cup \cdots \cup B_m| \geq km/k = m$. □

Proof of Sperner’s Lemma. Let $k = \max \{|A| : A \in S\}$. If $k \geq \lfloor n/2 \rfloor + 1$, let

$$\{A_1, \ldots, A_m\} = \{A \in S : |A| = k\}.$$

By the lemma, there are distinct sets $\phi(A_i)$ ($1 \leq i \leq m$), such that $\phi(A_i) \subset A_i$ and $|\phi(A_i)| = k-1$. Thus in $S$ we may replace each $A_i$ by $\phi(A_i)$ ($1 \leq i \leq m$), to obtain a new Sperner set $S'$ such that $|S'| = |S|$ and $A \in S' \Rightarrow |A| \leq k-1$. Repeating this process as many times as required, we obtain a Sperner set $S^*$ such that $|S^*| = |S|$ and $A \in S^* \Rightarrow |A| \leq \lfloor n/2 \rfloor$.

Now let $T = \{\{1, 2, \ldots, n\} \setminus A : A \in S^*\}$. Then $T$ is a Sperner set, $|T| = |S|$, and

$$A \in T \Rightarrow |A| \geq \lfloor n/2 \rfloor.$$

Thus, applying the above process to $T$, we obtain finally a Sperner set $T^*$ such that $|T^*| = |S|$ and $A \in T^* \Rightarrow |A| = \lfloor n/2 \rfloor$. Thus $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$.

References
