Behrend's Theorem for Sequences Containing No k-Element Arithmetic Progression of a Certain Type

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Abstract

Let k and n be positive integers, and let $d(n,k)$ be the maximum density in $\{0,1,2\ldots,k^n-1\}$ of a set containing no arithmetic progression of k terms with first term $a=\sum a_i k^i$ and common difference $d=\sum \varepsilon_i k^i$, where $0\le a_i\le k-1$, $\varepsilon_i=0$ or 1, and $\varepsilon_i=1\Rightarrow a_i=0$. Setting $\beta_k=\lim_{n\to\infty} d(n,k)$, we show that $\lim_{k\to\infty}\beta_k$ is either 0 or 1.

Throughout, we shall use the notation $[a,b)=\{a,a+1,a+2,\ldots,b-1\}$, for nonnegative integers $a<b$. Also, if S is a set of nonnegative integers, then $S(m)$ denotes $|S\cap[0,m)|$.

The upper asymptotic density of S will be denoted by $\bar{d}(S)$. Thus

$$\bar{d}(S)=\limsup_{m\to\infty} m^{-1}S(m).$$

Similarly, the lower asymptotic density of S is

$$\underline{d}(S)=\liminf_{m\to\infty} m^{-1}S(m).$$

Let $r_k(n)$ denote the largest cardinal of a subset A of $[0,n)$ such that A contains no arithmetic progression of k terms, and let $\rho_k=\lim_{n\to\infty} n^{-1}r_k(n)$. (This idea was introduced by Erdős, Turán, and Szekeres in [3], and then convergence of $n^{-1}r_k(n)$ is shown in [2].) K. F. Roth [6] proved $\rho_3=0$ in 1953 and E. Szemerédi [8] has shown that $\rho_k=0$ for all k.

Previous to these results, Felix Behrend [2] proved in 1937 that $\lim_{k\to\infty}\rho_k$ equals either 0 or 1. In this paper we prove the analogous result where ρ_k is replaced by β_k, the definition of β_k being similar to that of ρ_k except that only arithmetic progressions of a certain type are considered. (At the time of this writing, the only known values for β_k are $\beta_1=\beta_2=0$.) The main idea for the proof is taken diresctly from Behrend's paper.

Definition. For each positive integer k, a *k-diagonal* is an arithmetic progression on k terms with first term $a=\sum a_i k^i$ and common difference $d=\sum\varepsilon_i k^i$, where for each i, $0\le a_i\le k-1$, $\varepsilon_i=0$ or 1, and $\varepsilon_i=1\Rightarrow a_i=0$.

Note that k integers form a k-diagonal if and only if their k-ary representations can be put into the rows of a matrix in such a way that each column of the matrix, reading from top to bottom, is either $iii\cdots i$, for some i depending on the column, or $012\cdots k-1$. For example, $\{2,5,8\}$ is a 3-diagonal which contains no 2-diagonal.

Definition. For positive integers n,k, let

$$d(n,k) = k^{-n}|A|,$$

where A is a subset of $[0,k^n)$ which has largest cardinal while not containing any k-diagonal.

Thus for each fixed, k, we consider only the intervals $[0,k^n)$, $n = 1,2,\ldots,$, the reason for this is that we can think of $[0,k^n)$ as the set of all n-tuples on the k symbols $0,1,\ldots,k-1$, which seems to be an advantage.

The following lemma is proved in [1].

Lemma 1. *For each fixed k,* $\{d(n,k)\}_{n=1}^{\infty}$ *decreases. For each fixed n,* $\{d(n,k)\}_{k=1}^{\infty}$ *increases.*

Using this lemma, we can make the following definition.

Definition.

$$\beta_k = \lim_{n\to\infty} d(n,k), \qquad \text{for } k = 1,2,\ldots.$$
$$\beta = \lim_{k\to\infty} \beta_k.$$

Note that $0 \le \beta_1 \le \beta_2 \le \beta_3 \le \cdots \le \beta \le 1$.

As remarked earlier, our object is to prove that β is 0 or 1. We also remarked that the only currently known values of β_k are $\beta_1 = 0$ and $\beta_2 = 0$. The first follows directly from the definition of β_1. The second follows from observing that if $A \subset [0,2^n)$ then we may regard the elements of A (in binary notation) as characteristic functions of subsets of $\{1,2,\ldots,n\}$. It is then easy to see that A contains a 2-diagonal $\{x,y\}$ if and only if ther corresponding subsets X and Y of $\{1,2,\ldots,n\}$ satisfy $X \subset Y$ or $Y \subset X$. It then follows by Sperner's lemma that if A has no 2-diagonal then $|A| \le \binom{n}{\lfloor n/2 \rfloor}$, and therefore $d(n,2) = 2^{-n}\binom{n}{\lfloor n/2 \rfloor} \to 0$ as $n \to \infty$, that is, $\beta_2 = 0$.

Lemma 2. *If S is a set of nonnegative integers with upper density* $\bar{d}(S) > \beta_k$*, then S contains a k-diagonal.*

Proof. Let $\varepsilon > 0$ be such that $m^{-1}S(m) > \beta_k + \varepsilon$ for infinitely many m. Choose n so that $d(n,k) < \beta_k + \varepsilon/2$, and now choose m so that $\beta_k + \varepsilon < m^{-1}S(m)$ and $m^{-1}k^n < \varepsilon/2$. Finally, choose b so that $bk^n \le m < (b+1)k^n$. If S contains no k-diagonal, then in any interval $[ak^n,(a+1)k^n)$ S can have density at most $d(n,k)$, that is,

$$|S \cap [ak^n,(a+1)k^n)| \le k^n d(n,k).$$

Therefore $S(m) \le S(bk^n) + k^n \le bk^n d(n,k) + k^n$, hence

$$\beta_k + \varepsilon < m^{-1}S(m) \le d(n,k) + m^{-1}k^n < \beta_k + \varepsilon/2 + \varepsilon/2.$$

Therefore S contains a k-diagonal. □

Lemma 3. *For each k there is a set S with $\bar{d}(S) \geq \beta_k$ which does not contain a k^2-diagonal.*

Proof. Choose positive integers $n_1 < n_2 < \cdots$ such that $n_{i+1} - n_i \to \infty$. For each i, let $A_i \subset [0, k^{n_i})$ be such that A_i contains no k-diagonal and $|A| = k^{n_i} d(n_i, k)$. Let $B_i = A_i \cap [2k^{n_{i-1}}, k^{n_i})$, and let $S = \bigcup_{i=1}^{\infty} B_i$. Then $k^{-n_i} S(k^{n_i}) \geq k^{-n_i}(k^{n_i} d(n_i, k) - 2k^{n_{i-1}}) \to \beta_k$ (we may assume $k \geq 2$ since $\beta_1 = 0$), hence S has upper density β_k.

Now because of the size of the gaps between successive blocks B_i, no arithmetic progression can intersect more than two of the B_i's. In particular, if S contains a k^2-diagonal D, then either the first k elements of D belong to some B_i, or the last k elements of D belong to some B_j (or both). But the first k elements of a k^2-diagonal constitute a k-diagonal, and similarly for the last k elements. Since no B_i contains a k-diagonal, S can contain no k^2-diagonal. □

Lemma 4. *(a) If $D = \{a_i : i \in [0, k^{pn})\}$ is a k^{pn}-diagonal and $J \subset [0, k^{pn})$ is a k^n-diagonal then $D' = \{a_j : j \in J\}$ is a k^n diagonal.*

(b) If D is a k^n-diagonal and $\ell \in [0, k^n)$ is fixed, then $D' = \{k^n a + \ell : a \in D\}$ is a k^n-diagonal.

Proof. (a) Express each element of D in k^{pn}-ary notation, so that $D = \{X_1 i X_2 i \cdots i X_d : i \in [0, k^{pn})\}$, where each X_j is a block (possibly empty) of k^{pn}-ary symbols and i is a single k^{pn}-ary symbol running from 0 to $k^{pn} - 1$. Replacing each k^{pn}-ary symbols by its equivalent string of p $k^{pn} - 1$-ary symbols, we obtain $D = \{X_1' i' X_2' i' \cdots i' X_d' : i \in [0, k^{pn})\}$, where each X_j' is a block of k^n-ary symbols and i' is a block of p k^n-ary symbols running from 0 to $k^{pn} - 1$. It is now clear that if J is a k^n-diagonal contained in $[0, k^{pn})$ then so is $D' = \{X_1' j X_2' j \cdots j X_d' : j \in J\}$.

(b) If $D = \{X_1 i X_2 i \cdots i X_d : i \in [-, k^n)\}$ (each element of D is expressed in k^n-ary notation), then $D' = \{X_1 i X_2 i \cdots i X_d \ell : i \in [0, k^n)\}$. □

The following lemma is proved in [4] and [5].

Lemma 5. *If k, c are positive integeres then there is an integer $N(k, c)$ such that if $m \geq N(k, c)$ and $[0, m)$ is partitioned in any way into c classes, then at least one class contains a k-diagonal.*

We are now ready to prove the main theorem.

Theorem. *β equals 0 or 1.*

Proof. Suppose $0 < \beta < 1$, and choose k so that $\beta_k \cdot (1/\beta) > \beta_{k^2}$. Next choose $\varepsilon > 0$ so that $\varepsilon < \frac{1}{4}(\beta_k - \beta \beta_{k^2})$, and using Lemma 3 let S be a set of nonnegative integers with $\bar{d}(S) \geq \beta_k$ which contains no k^2-diagonal. Next, choose n large enough that if $A \subset [0, k^{2n})$ and $|A| > (\beta_{k^2} + \varepsilon) k^{2n}$ then A must contain a k^2-diagonal. For each $j = 0, 1, \ldots$ let $B_j = S \cap [jk^{2n}, (j+1)k^{2n})$. We now partition the nonnegative integers j into $2^{k^{2n}}$ classes as follows. For each $\sigma \subset [0, k^{2n})$, j belongs to the class C_σ if and only if B_j is a translate of σ.

There are now two main steps in the proof. The first is to show that $\bar{d}\left(\bigcup_{\sigma \neq \phi} C_\sigma\right) > \beta$; the second is, using this, to extract a k^2-diagonal from S, contrary to our initial assumption.

To show that $\bar{d}\left(\bigcup_{\sigma \neq \phi} C_\sigma\right) > \beta$, we will show that $\underline{d}(C_\phi) < 1 - \beta$. Let $\underline{d}(C_\phi) = \xi$, and choose m so that $C_\phi(m) > (\xi - \varepsilon) m$ and $S(mk^{2n}) > (\beta_k - \varepsilon) mk^{2n}$. Then $\left(\bigcup_{\sigma \neq \phi} C_\sigma\right)(m) < (1 - \xi + \varepsilon) m$, and for every j, $|B_j| \leq (\beta_{k^2} + \varepsilon) k^{2n}$.

Hence

$$S(mk^{2n}) = \left| \bigcup \left\{ B_j : j \in [0,m) \cap \left(\bigcup_{\sigma \neq \phi} C_\sigma \right) \right\} \right|$$
$$< (\beta_{k^2} + \varepsilon) k^{2n} \cdot (1 - \xi + \varepsilon) m,$$

hence

$$\beta_k - \varepsilon < (\beta_{k^2} + \varepsilon)(1 - \xi + \varepsilon),$$

$$\xi \leq [(1+\varepsilon)(\beta_{k^2} + \varepsilon) - \beta_k + \varepsilon]/(\beta_{k^2} + \varepsilon)$$
$$< (\beta_{k^2} - \beta_k + 4\varepsilon)/\beta_{k^2} < 1 - \beta,$$

by the choice of ε. This completes the first step.

For the final step in the proof, choose by Lemma 5 an integer p large enough that if $[0, k^{2pn})$ is partitioned into $2^{k^{2n}-1}$ classes, then at least one class will contain a k^{2n}-diagonal. Since we now know that $\bar{d}\left(\bigcup_{\sigma \neq \phi} C_\sigma\right) > \beta \geq \beta_{k^{2pn}}$, it follows from Lemma 2 that $\bigcup_{\sigma \neq \phi} C_\sigma$ contains a k^{2pn}-diagonal $D = \{a_i : i \in [0, k^{2pn})\}$. Let us now partition the indices i of the elements of D into classes C'_σ according to the rule $i \in C'_\sigma \Leftrightarrow a_i \in C_\sigma$. Then by the choice of p there is a k^{2n}-diagonal $J \subset [0, k^{2pn})$ which is contained in a single class, say $J \subset C'_{\sigma_0}$. This means that $D' = \{a_j : j \in J\}$. is contained in C_{σ_0}. But by Lemma 4(a), D' is a k^{2n}-diagonal. Thus we now have that the k^{2n} sets $B_{a_j} = S \cap [a_j k^{2n}, (a_j+1)k^{2n})$, $j \in J$, are all translates of σ_0, and $D' = \{a_j : j \in J\}$ is a k^{2n}-diagonal. Since $\sigma_0 \neq \phi$, we may choose $\ell \in \sigma_0$. Then $D'' = \{k^{2n} a_j + \ell : j \in J\} \subset S$, and by Lemma 4(b) D'' is a k^{2n}-diagonal. But then the first k^2 elements of this k^{2n}-diagonal are a k^2-diagonal in S, contrary to assumption.

Thus the proof is complete. □

References

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