A coloring of the non-negative integers, with applications

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Abstract

We describe a particular partition of the non-negative integers which consists of infinitely many translates of an infinite set. This partition is used to show that a certain van der Waerden-like theorem has no simple canonical version. The partition is also used to give a lower bound for one of the classical van der Waerden functions, namely $w(3;m)$, the smallest positive integer such that every m-coloring of $[1,w(3;m)]$ produces a monochromatic 3-term arithmetic progression. Several open questions are mentioned.

Introduction

Let S denote the set of all distinct sums of odd powers of 2, including 0 as the empty sum. Then every non-negative integer can be written uniquely in the form $s+t$, where $s \in S$ and $t \in T$, and define $f(n) = s$. In other words, if $n = \sum_{i \text{ odd}} 2^i + \sum_{i \text{ even}} 2^i$, then $f(n) = \sum_{i \text{ odd}} 2^i$. For this coloring f, the set of colors is S, and for each $s \in S$, f is constant on the "color class" $s+T$.

A van der Waerden-like theorem, and its canonical version

We need the following definition.

Definition 1. *If* $A = \{a_1 < a_2 < \cdots < a_n\} \subset \omega = \{0,1,2,\ldots\}$, $n \geq 2$, *the* gap size *of* A *is* $\mathrm{gs}(A) = \max\{a_{j+1} - a_j : 1 \leq j \leq n-1\}$. *If* $|A| = 1$, $\mathrm{gs}(A) = 1$.

Theorem 1. *If* ω *is finitely colored, there exist a fixed* $d \geq 1$ *(*d *depends only on the coloring) and arbitrarily large (finite) monochromatic sets* A *with* $\mathrm{gs}(A) = d$.

This fact first appeared in [3]. A proof can be found in [12]. Various applications appear in [4, 6, 11, 13].

Theorem 1 is somewhat similar in form to van der Waerden's theorem on arithmetic progressions [15]. (Van der Waerden's theorem says that for every k, every finite coloring of the positive integers produces a monochromatic k-term arithmetic progression.) However, Theorem 1 differs in a number of ways:

Van der Waerden's theorem does not imply Theorem 1, since the d in the conclusion of Theorem 1 is independent of the size of the monochromatic sets A. Beck [1] showed the existence of a 2-coloring of ω such that if A is any monochromatic arithmetic progression with common difference d, then $|A| < 2\log d$. Hence the presence of large monochromatic arithmetic progressions, which is guaranteed by van der Waerden's theorem, is not enough to imply Theorem 1. Somewhat earlier, Justin [10] found an explicit coloring such that if A is any monochromatic arithmetic progression with common difference d, then $|A| < h(d)$; in his example, the coloring is explicit but the function $h(d)$ is not.

Theorem 1 (which has a simple proof) does not imply van der Waerden's theorem in a simple way. (In Chapter 14 of [8], Hindman and Strauss give a proof that Fact 1 does in fact imply van der Waerden's theorem – and at this point in their book, the proof *does* seem simple – however, a fair amount of machinery has been developed by this point.)

Theorem 1 does not have a density version corresponding to Szemerédi's theorem [14]. That is, there exists a set $X \subset \omega$ with positive upper density for which there do *not* exist a fixed $d \geq 1$ and arbitrarily large sets $A = \{a_1 < a_2 < \cdots < a_n\} \subset X$ with $\mathrm{gs}(A) = d$. For an example of such a set X, see [2].

Finally, no "canonical version" of this result is known. The Erdős-Graham canonical version of van der Waerden's theorem ([7]) states that if $g : \omega \to \omega$ is an arbitrary coloring of ω (using finitely many or infinitely many colors) then there exist arbitrarily large arithmetic progressions A such that either g is constant on A, i.e. $|g(A)| = 1$, or g is one-to-one on A, i.e. $|g(A)| = |A|$.

We show that there is no such canonical version of Theorem 1. This is Corollary 1 below.

A very brief sketch of an outline of a proof of the following result has appeared in [5]. It seems worthwhile to fill in some of the missing details.

Theorem 2. *For every $A \subset \omega$ (with f as described in the introduction),*

$$\frac{1}{4}\sqrt{|A|/\mathrm{gs}(A)} < |f(A)| < 4\sqrt{|A|\mathrm{gs}(A)}$$

Corollary 1. *For the coloring f above, there do not exist a fixed d and arbitrarily large sets A with $\mathrm{gs}(A) = d$ on which f is either constant or 1-1.*

Proof of Corollary 1. If $16\mathrm{gs}(A) \leq |A|$, then by Theorem 2, $1 < |f(A)| < |A|$. □

To prove Theorem 2, we need the following definition.

Definition 2. *For $k \geq 0$, an* aligned block *of size 4^k is a set of 4^k consecutive non-negative integers whose smallest element is $m4^k$, for some $m \geq 0$.*

Proof of Theorem 2. Note that the first aligned block of size 4^k, namely $[0,4^k-1]=[0,2^{2k}-1]$, is in 1-1 correspondence with the set of all binary sequences of length $2k$. From this we see (by the definition of f) that for $n\in[0,2^{2k}-1]$, there are 2^k possible values of $f(n)$, and each value occurs exactly 2^k times. It is easy to see (using the definition of f) that the same is true for any aligned block $[m4^k, m4^k+4^k-1]$. We express this more simply by saying that *"each aligned block of size 4^k has 2^k colors, each appearing exactly 2^k times."*

Now we can establish the upper bound in Theorem 2. Let $A=\{a_0<a_1<a_2<\cdots<a_n\}\subset\omega$. Then $a_n\le a_0+n\mathrm{gs}(A)=a_0+(|A|-1)\mathrm{gs}(A)$, or

$$a_n-a_0<|A|\mathrm{gs}(A).$$

Choose s minimal so that A is contained in the union of two adjacent aligned blocks of size 4^s. (Two blocks are necessary in case A contains both $m4^s-1$ and $m4^s$ for some m.) Then

$$4^{s-1}<a_n-a_0.$$

Since each aligned block of size 4^s has 2^s colors,

$$|f(A)|\le 2\cdot 2^s.$$

Putting these three inequalities together gives

$$|f(A)|<4\sqrt{|A|\mathrm{gs}(A)}.$$

Next, we establish the lower bound for $|f(A)|$, which requires a bit more care. We will use the following Lemma.

Lemma 1. *For each $k\ge 0$, an two aligned blocks of size 4^k (consecutive or not) are either colored identically, or have no color in common.*

Proof of Lemma 1. Consider the aligned blocks $[p4^k,p4^k+4^k-1]$ and $[q4^k,q4^k+4^k-1]$. By the definition of f (and since 4^k is an even power of 2), $f(p4^k)=f(p)4^k$, so that $f(p4^k)=f(q4^k)$ if and only if $f(p)=f(q)$. Also, for $0\le j\le 4^k-1$, $f(p4^k+j)=f(p4^k)+f(j)$. This last equality obviously holds if $p=0$, and for $p>0$ it holds since then each power of 2 which occurs in j is less than each power of 2 which occurs in $p4^k$. Thus the blocks $[p4^k,p4^k+4^k-1]$ and $[q4^k,q4^k+4^k-1]$ are colored identically if $f(p)=f(q)$, and have no color in common if $f(p)\ne f(q)$.

Proceeding with the lower bound in Theorem 2, we note that for $k\ge 1$, the colors of any aligned block of size 4^k have the form $UUVV$, where U and V are blocks of size 4^{k-1}.

Next, we note that any block of size 4^k, aligned or not, contains *at least* 2^k colors. for let A be any block of size 4^k. Let the first element of A lie in the aligned block S of size 4^k, and let T be the aligned block of size 4^k which immediately succeeds S. If S and T are colored identically, then the elements of $f(A)$ are just a cyclic permutation of the elements of $f(S)$, and hence the block A contains exactly 2^k colors. By Lemma 1, the remaining case is when S,T have no color in common. In this case, by the preceding paragraph, $f(S)f(T)=UUVVXXYY$, where no two of U,V,X,Y have a color in common, and so U, V, X, Y are of size 4^{k-1}. Then $f(A)$, which has size 4^k, contains either UV or VX or XY, and so has at least $2^{k-1} + 2^{k-1} = 2^k$ colors.

Finally, we note that for $s \geq 1$, $k \geq 1$, every set of 4^s consecutive aligned blocks of size 4^k contains at least 2^s blocks of size 4^k, no two of which have a common color. This follows from the fact that these p^s blocks have the form $[p4^k, p4^k + 4^k - 1]$, $t \leq p \leq t + 4^s - 1$, for some t. The block $f\ ([t, t+4^s-1])$ has at least 2^s colors, by the preceding paragraph. If $f(p) \neq f(q)$, where $t \leq p < q \leq t + 4^s - 1$, then $f(p4^k) \neq f(q4^k)$, so by Lemma 1 the two blocks $[p4^k, p4^k + 4^k - 1]$ and $[q4^k, q4^k + 4^k - 1]$ have no color in common.

Now let $A \subset \omega$ be given. Choose k so that $4^{k-1} \leq \mathrm{gs}(A) < 4^k$. Choose t minimal so that A is contained in the union of t consecutive aligned blocks of size 4^k. Then A meets each of these blocks (by the choice of k), and

$$|A| \leq t4^k.$$

Choose s so that $4^s \leq t < 4^{s+1}$. Then among the t consecutive aligned blocks of size 4^k are at least 2^s blocks of size 4^k, no two of which have a color in common. Since each of the t blocks meets A, we have

$$2^s \leq |f(A)|.$$

Thus $|A| \leq t4^k < 4 \cdot 4^s \cdot 4 \cdot 4^{k-1} \leq 4|f(A)|^2 \cdot 4 \cdot \mathrm{gs}(A)$, so $\frac{1}{4}\sqrt{|A|/\mathrm{gs}A} < |f(A)|$. □

□

A bound for a van der Waerden function

Definition 3. *For $m \geq 1$, let $w(3;m)$ denote the smallest positive integer such that every m-coloring of $[1, w(3;m)]$ produces a monochromatic 3-term arithmetic progression.*

Theorem 3. *For all $m \geq 1$, $w(3;m) > \frac{1}{2}m^2$.*

Proof. For $k \geq 1$, the coloring f described in the introduction colors the interval $[0, 2^{2k+1} - 1]$ with 2^k colors. The colors are the sums (including 0 as the empty sum) of distinct elements of the set $\{2^1, 2^3, 2^5, \ldots, 2^{2k-1}\}$. The color classes are subsets of the translates (by the 2^k colors) of the set S_k of sums (including 0 as the empty sum) of distinct elements of the set $\{2^0, 2^2, 2^4, \ldots, 2^{2k}\} = \{4^0, 4^1, 4^2, \ldots, 4^k\}$. It is easy to see that S_k contains no 3-term arithmetic progression. Hence, with respect to the coloring f, there is no monochromatic 3-term arithmetic progression in $[0, 2^{2k+1} - 1]$. The coloring f shows that for $k \geq 1$, $w(3; 2^k) > 2^{2k+1}$. For a general m, choose k so that $2^k \leq m < 2^{k+1}$. Then $w(3;m) \geq w(3;2^k) > 2^{2k+1} = \frac{1}{2}2^{2k+2} < \frac{1}{2}m^2$. □

Remarks

1. The lower bound in Theorem 3 is not the best possible. Indeed, in the standard reference Ramsey Theory (by R. L. Graham, B. L. Rothschild, and J. H. Spencer, 2nd edition, 1990, John Wiley & Sons, New York), the authors show that for some positive constant c, $w(3;m) > m^{(c \log m)}$.

2. Of course, one would like to have an upper bound for the function $w(3;m)$. The only bound known to me is $w(3;m) < \left(\frac{m}{4}\right)^{3^m}$ for $m > 4$. This bound comes from [9], and is mentioned in [12]. The coloring f on $[0, 2^{2k+1}-1]$, with 2^k colors, is perhaps "efficient" in stopping all monochromatic 3-term arithmetic progressions. Cutting the number of colors in half would seem to leave too few colors. If this were in fact true, then $w(3;2^{k-1}) \leq 2^{2k+1}$ would follow, and for general m one would then have $\frac{1}{2}m^2 < w(3;m) < 8m^2$.

3. Corollary 1 shows that a constant/1-1 canonical version of Theorem 1 is not true. We also know by the Bergelson/Hindman/McCutcheon example that a density version of Theorem 1 is not true.

   The following three simple examples, involving only 3-element sets, illustrate various combinations of the truth or falsity of the "constant/1-1 versions" and the "density versions."

   (a) The simplest non-trivial case of van der Waerden's theorem says that every finite coloring of the positive integers produces a monochromatic 3-term arithmetic progression. The constant/1-1 version of this reselt holds by the Erdős-Graham theorem, and the density version holds by Szemerédi's theorem.

   (b) Schur's theorem says that if the positive integers are finitely colored, then there is a monochromatic solution of $x+y=z$. The density version does not hold by taking all the odd integers. The constant/1-1 version does not hold by coloring each x with the highest power of 2 dividing x.

   (c) At the meeting, Kevin O'Bryant showed me this example: if the positive integers are finitely colored, then there is a monochromatic 3-term geometric progression (a set of the form $\{a, ad, ad^2\}$). To get the constant/1-1 version, let a coloring g of the positive integers be given. define a new coloring h by setting $h(x) = g(2^x)$. Then, by the Erdős-Graham theorem, there is a set $\{a, a+d, a+2d\}$ on which the coloring h is either constant or 1-1. The density version does not hold, since the set of square-free numbers has positive density.

   (d) It seems natural to ask for a collection P of 3-element sets (if such a collection exists!) for which: (i) Every set of positive integers with positive upper density contains an element of P; (ii) It's not the case that for every coloring of the positive integers, there is an element of P on which the coloring is either constant or 1-1.

4. It would be nice to be able to say *something* about general colorings along the lines of Theorem 1. Perhaps the following is true: if $\omega \to \omega$ is an arbitrary coloring of ω, then there exist a fixed $d \geq 1$ and arbitrarily large (finite) sets A with $\mathrm{gs}(A) = d$ such that either

   (a) at most $\sqrt{|A|}$ distinct colors appear in $g|_A$; or

   (b) each color appears in $g|_A$ at most $\sqrt{|A|}$ times.

   Note that for the particular coloring f, if we take $d=1$, and let $A = [0, 4^k - 1]$, then exactly $\sqrt{|A|}$ distinct colors appear in $f|_A$, and each color appears in $f|_A$ exactly $\sqrt{|A|}$ times.

5. We have used a particular partition of ω. We would get another partition of ω (into infinitely many translates of an infinite set) by replacing the odd powers of 2 and the even powers of 2 by arbitrary

A and B, where $\{A,B\}$ is any partition of $\{1,2,3,\dots\}$ into two infinite sets. Perhaps it's possible to describe *all* of the partitions of ω into infinitely many translates of an infinite set.

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