On Locally Finite Semigroups

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Citation data: T.C. Brown, *On locally finite semigroups* (In Russian), Ukraine Math. J. **20** (1968), 732–738.

Abstract

The classical theorem of Schmidt on locally finite group extensions may be stated as follows: If $\varphi : G \mapsto H$ is a homomorphism of the group G onto the locally finite group H with locally finite kernel, then G is locally finite.

In this paper we prove the exact analogue of this theorem for semigroups. Then in the last section we give several consequences, including the well-known theorem of Shevrin which states that a band of locally finite semigroups is locally finite, and the theorem of Green and Rees on the equivalence of Burnside's problem for groups with "Burnside's problem for semigroups".

1 Introduction

The whole of this paper is based on Lemma 2 below, which is essentially combinatorial. The methods are completely elementary in nautre, and the paper is self-contained, except for a few facts used in the last two sections. The theorem we wish to prove is the following.

Theorem. *If $\varphi : S \mapsto T$ is a homomorphism of the semigroup S onto the locally finite semigroup T such that $e\varphi^{-1}$ is a locally finite subsemigroup of S for each idempotent element e of T, then S is locally finite.*

First we note that it is sufficient to consider the case where T is finite. For suppose the theorem is true in this case, and let $\varphi : S \mapsto T'$ be a homomorphism with all the required properties onto an arbitrary (possibly infinite) locally finite semigroup T'. Let A be a finite subset of S, and let $\langle A\rangle$ denote the subsemigroup of S generated by A. It is required to show that $\langle A\rangle$ is finite. Now $\langle A\varphi\rangle = T$ is a finite subsemigroup of T', since T' is locally finite, hence all we have to do is to restrict φ to $T\varphi^{-1}$ to get a homomorphism $\varphi' : T\varphi^{-1} \mapsto T$ onto a finite semigroup T; furthermore φ' has all the required properties. Hence by our assumption, $T\varphi^{-1}$ is locally finite. But $A \subset T\varphi^{-1}$, hence $\langle A\rangle$ is finite, as required.

In Sections 2 throug 4 below, we shall prove the theorem for the special cases where T is a (finite) group, group with zero, null semigroup, simple semigroup, or 0-simple semigroup. (For definitions see below). In Section 5 we give some consequences to the theorem.

Assuming the truth of the theorem for the special cases listed above, the general case follows by induction on $|T|$, the order of T, as follows. For $|T| = 1$, the theorem is trivial. Let T be a finite semigroup and suppose the theorem holds for all semigroups T' with $|T'| < |T|$. If T has no proper non-zero ideals, then T is either null, simple, or 0-simple, and S is then locally finite by the appropriate special case. If T has a proper non-zero ideal M, then let $T' = T/M$, the Rees factor semigroup (for definition, see below), let ψ be the natural mapping of T upon T', and let $\sigma = \varphi\psi$. Since $\sigma$is a homomorphism of S upon T', and $|T'| < |T|$, all we have to verify is that $e\sigma^{-1}$ is locally finite for each idempotent element e of T'; for then by the inductive assumption S is locally finite. Thus let e be an idempotent element of T'. If $e \neq 0$, then $e\sigma^{-1} = (e\psi^{-1})\varphi^{-1} = e\varphi^{-1}$, and $e\varphi^{-1}$ is locally finite by hypothesis. If $e = 0$, then $0\sigma^{-1} = (0\psi^{-1})\varphi^{-1} = M\varphi^{-1}$, and $M\varphi^{-1}$ is locally finite by the inductive assumption.

In the remainder of the proof (and above), the following definitions are used. If a semigroup S contains an element 0 such that $0s = s0 = 0$ for all $s \in S$, S is a *semigroup with zero*. In case $S \setminus \{0\}$ is a group, S is a *group with zero*. A subsemigroup M of S is an *ideal* is $SMS \cup MS \cup SM \subset M$, and is a *proper ideal* if $M \neq S$. A semigroup without proper ideals is *simple*. A semigroup S with zero is *0-simple* if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper ideal of S. S is *null* if $S^2 = \{0\}$. Let M be an ideal of S; the *Rees factor semigroup of S modulo M*, denoted by S/M, is defined as follows. As a set, $S/M = (S \setminus M) \cup \{0\}$. For $x, y \in S$, let xy denote their product in S and $x * y$ their product in S/M. Then, by definition, $x * y = 0$ if $xy \in M$, $x * y = xy$ if $xy \notin M$, and $0 * z = z * 0 = 0$ for all $z \in S/M$. The mapping $x \mapsto 0$, $x \in M$, $x \mapsto x$, $x \notin M$, is the *natural homomorphism* of S upon S/M.

We shall also require the following:

Let A be a subset of the semigroup S. If $x \in \langle A \rangle$, then we say that x has *length m* (with respect to A) if x can be written as the product of m elements (not necessarily distinct) from A and cannot be written as the product of any smaller number of elements from A. Thus, x has length m if and only if

(1) $x = x_1 x_2 \cdots x_m$ ($x_i \in A, 1 \leq i \leq m$)

(2) $x = y_1 y_2 \cdots y_n$ ($y_i \in A, 1 \leq i \leq n$) implies $n \geq m$.

If x has length m with respect to some set A, we simply write $|x| = m$; the paricular set A upon which $|x|$ depends will be clear from the context.

If $x, y \in \langle A \rangle$ and $|x| + |y| = |xy|$, we say that x is a *left segment* of xy. (Here we do *not* allow $|x| = 0$.) If $x, y, z \in \langle A \rangle$ and $|x| + |y| + |z| = |xyz|$, we say that y is a *segment* of xyz. (Here we *do* allow $|x| = 0$ or $|z| = 0$.)

2

In this section we prove the theorem for the case where T is a finite group. The proof is by a rather curious contradiction. We deny the theorem and then apply a kind of sieve to produce an element whose image under φ is not in the range of φ.

Lemma. *Let T be a group with identity e and let $\varphi : S \mapsto T$ be a homomorphism of the semigroup S onto T such that $e\varphi^{-1}$ is locally finite. Let A be any finite subset of S and let m be any positive integer. Then there exists an integer $k = k(m, A)$ with the following properties: If $x \in \langle A \rangle$, $|x| = km$ (length with repect to A), and $g \in T$, then $x = P(g)Q(g)R(g)$, where $P(g), Q(g), R(g) \in \langle A \rangle$, $|Q(g)| = m$, and $(P(g)U)\varphi \neq g$ for every left segment U of $Q(g)$.*

Proof. Let $x \in \langle A \rangle$, $|x| = km$, $x = A_1 A_2 \cdots A_k$, $A_i \in \langle A \rangle$, $|A_i| = m$, $1 \leq i \leq k$, $g \in T$. We will show that if k is taken large enough, then some A_i may be chosen as $Q(g)$. For suppose the contrary. Then for arbitrarily large k we can find $x \in \langle A \rangle$ such that $|x| = km$, $x = A_1 A_2 \cdots A_k$, $A_i \in \langle A \rangle$, $|A_i| = m$, $A_i =$

B_iC_i, $|B_i| + |C_i| = m$, $1 \le i \le k$, and $(B_1)\varphi = (B_1C_1B_2)\varphi = \cdots = (B_1C_1\cdots B_{k-1}C_{k-1}B_k)\varphi = g$. Then $C_1B_2, C_2B_3, \ldots, C_{k-1}B_k \in e\varphi^{-1}$, and since $|B_i| + |C_i| = m$, we have $|C_iB_{i+1}| \le 2m$.

Now let H be the subsemigroup of $e\varphi^{-1}$ generated by the finite set

$$\{y \in \langle A\rangle \cap e\varphi^{-1} : |y| \le 2m\}.$$

Then H is finite since $e\varphi^{-1}$ is locally finite. Furthermore, H depends only on A and m. Thus there exists a number $M = M(A,m)$ such that

$$z \in H \Rightarrow |z| \le M.$$

(Note we are still writing lengths with respect to the set A.)

The point of constructing H is that

$$W = C_1B_2C_2B_3\cdots C_{k-1}B_k \in H,$$

hence $|W| \le M$. But then

$$km = |x| = |B_1WC_k| \le |B_1| + |W| + |C_k| \le M + 2m,$$

or $(k-2)m \le M$.

Since M does not depend on k, this is a contradiction for sufficiently large k. This proves Lemma 2.

□

Lemma. *Let T be a group with identity e, and let $\varphi : S \mapsto T$ be a homomorphism of the semigroup S onto T such that $e\varphi^{-1}$ is locally finite. Let $V \subset T$, $g \in T$, $g \notin V$. Let A be a finite subset of S, and suppose that $x \in \langle A\rangle$, $x = P_1Q_1R_1$, where $Q_1 \in \langle A\rangle$, $|Q_1| = k(m,A).m$ (see Lemma 2), and $(P_1U_1)\varphi \notin V$ for every left segment U_1 of Q_1.*

Then $x = P_1P_2Q_2R_2R_1$, where $Q_2 \in \langle A\rangle$, $|Q_2| = m$, and $(P_1P_2U_2)\varphi \notin V \cup \{g\}$ for every left segment U_2 of Q_2.

Proof. Let $h = (P\varphi)^{-1}g$. Since $|Q_1| = k(m,A).m$, by Lemma 2 we have $Q_1 = P_2Q_2R_2$, where $Q_2 \in \langle A\rangle$, $|Q_2| = m$, and $(P_2U_2)\varphi \ne h$ for every left segment U_2 of Q_2. Now if U_2 is a left segment of Q_2, then P_2U_2 is a left segment of Q_1, therefore by the hypotheses of the present lemma $(P_1P_2U_2)\varphi \notin V$. But also $(P_1P_2U_2)\varphi \ne g$, for otherwise we would have $(P_2U_2)\varphi = (P_1\varphi^{-1})g = h$, a contradiction. Therefore $(P_1P_2U_2)\varphi \notin V \cup \{g\}$, as required. □

Lemma. *Let T be a finite group with identity e, and let $\varphi : S \mapsto T$ be a homomorphism of the semigroup S onto T such that $e\varphi^{-1}$ is locally finite. Then S is locally finite.*

Proof. Let T have n elements $\{g_1, \ldots, g_n\}$. Let A be a finite subset of S. In the notation of Lemma 2, let $k_0 = 1, k_1 = k(k_0, A)$, $k_2 = k(k_0k_1, A), \ldots,$ $k_n = k(k0k_1\cdots k_{n-1}, A)$. We shall show that $x \in \langle A\rangle$ implies $|x| < k_0k_1\cdots k_n$. This of course means that $\langle A\rangle$ is finite, and so S is locally finite.

To prove our assertion, suppose $x \in \langle A\rangle$, $|x| \ge k_0k_1\cdots k_{n-1}k_n$. We may as well assume that $|x| = k_0k_1\cdots k_{n-1}k_n$. Then

$$|x| = k_n.(k_0k_1\cdots k_{n-1}) = k(k_0k_1\cdots k_{n-1}, A).(k_0k_1\cdots k_{n-1}),$$

so by Lemma 2 $x = P_1Q_1R_1$, where $Q_1 \in \langle A\rangle$, $|Q_1| = k_0k_1\cdots k_{n-1}$, and $(P_1Q_1)\varphi \neq g$ for every left segment U_1 of Q_1.

Now suppose we have $x = P_1\cdots P_mQ_mR_m\cdots R_1$, where $Q_m \in \langle A\rangle$, $|Q_m| = k_0k_1\cdots k_{n-m} = k_{n-m}.(k_0k_1\cdots k_{n-m-1}) = k(k_0k_1\cdots k_{n-m-1}, A).(k_0k_1\cdots k_{n-m})$, and

$$(P_1\cdots P_mU_m)\varphi \notin \{g_1,\ldots,g_m\}$$

for every left segment U_m of Q_m.

We now use Lemma 2 to sieve out the element g_{m+1}, and obtain

$$x = P_1\cdots P_{m+1}Q_{m+1}R_{m+1}\cdots R_1,$$

where $Q_{m+1} \in \langle A\rangle$, $|Q_{m+1}| = k_0k_1\cdots k_{n-m-1}$, and $(P_1\cdots P_{m+1})\varphi \notin \{g_1,\ldots,g_{m+1}\}$ for every left segment U_{m+1} of Q_{m+1}.

Thus after n steps we have $x = P_1\cdots P_nQ_nR_n\cdots R_1$, where $Q_n \in \langle A\rangle$, $|Q_n| = k_0 = 1$ (so that Q_n has exactly one left segment, namely Q_n), and $(P_1\cdots P_nQ_n)\varphi \notin \{g_1,\ldots,g_n\} = T$. Since φ is after all a mapping of S into T, this is a contradiction, and completes the proof. □

3

We now consider the cases wherer T is either a finite group with zero or a finite null semigroup.

Let S, φ, T be as in the theorem, and suppose also that T is a finite group with zero. Let $A = (T \setminus \{0\})\varphi^{-1}$; then A is locally finite by Lemma 2. Also, by assumption, $B = 0\varphi^{-1}$ is locally finite. Thus we have $S = A \cup B$, where A, B are locally finite and B is an ideal in S. It is easy to see in this case that S is locally finite. In the case that T is a finite null semigroup, again letting $B = 0\varphi^{-1}$, we have that B is a locally finite ideal in S and $S^2 \subset B$. Here again it is easy to see that S is locally finite. We summarize these cases as

Lemma. *Let T be either a finite group with zero or a finite null semigroup, and let $\varphi : S \mapsto T$ be a homomorphism of the semigroup S onto T such that $e\varphi^{-1}$ is locally finite for each idempotent element e of T. Then S is locally finite.*

4

In this section we consider the remaining cases, where T is either a (finite) simple semigroup or 0-simple semigroup.

Let T be a finite simple of finite 0-simple semigroup. Then T is completely simple or completely 0-simple, and it follows in a standard way ([3]) that if $a, b \in T$ and $TabT \neq \{0\}$, then bTa is a finite group or a finite group with zero. This fact will be used in what follows.

The next lemma is a variation on Lemma 2. Its proof is briefly sketched.

Lemma. *Let T be a finite simple or finite 0-simple semigroup, and let $\varphi : S \mapsto T$ be a homomorphism or the semigroup S onto T such that $e\varphi^{-1}$ is locally finite for each idempotent element e of T. Let A be* *any finite subset of S and let m be any positive integer. Let a, b be element of A such that $|ab| = 2$ and $T((ab)\varphi)T \neq \{0\}$. Then there exists an integer $k = k(ab, m, A)$ with the following properties: If $x \in \langle A \rangle$, $|x| = km$, then $x = PQR$ where $Q \in \langle A \rangle$, $|Q| = m$, and ab is not a segment of Q.*

Proof. Assume the contrary. Then for arbitrarily large k we can find $x \in \langle A \rangle$ such that $|x| = km$, $x = A_1A_2 \cdots A_k$, $A_i \in \langle A \rangle$, $|A_i| = m$, $A_i = B_1abC_i$, $|B_i| + 2 + |C_i| = m$, $1 \leq i \leq k$. Thus $x = B_1aybC_k$, where

$$y = \prod_{i=1}^{k-1} (bC_iB_{i+1}a).$$

Let $G = (b\varphi)T(a\varphi)$. Then G is a finite group or a finite group with zero, and so $G\varphi^{-1}$ is locally finite by Lemmas 2 and 3. But y belongs to a finitely generated subsemigroup of $G\varphi^{-1}$, hence $|y|$ is bounded above by a number which depends only on a, b, m, and A, hence $|x|$ is similarly bounded. For sufficiently large k this contradicts $|x| = km$. □

Lemma. *Let T be a finite simple or finite 0-simple semigroup. Let $\varphi : S \mapsto T$ be a homomorphism of the semigroup S onto T such that $e\varphi^{-1}$ is locally finite for each idempotent element e of T. The S is locally finite.*

Proof. Let A be a finite subset of S. Let

$$\begin{aligned} B &= \{xy : x, y \in A, |xy| = 2, T((xy)\varphi)T \neq \{0\}\}, \\ C &= \{xy : x, y \in A, |xy| = 2, T((xy)\varphi)T = \{0\}\}, \\ D &= \{a_1b_1, \ldots, a_pb_p\}. \end{aligned}$$

(Note that if T is simple then C is empty.)

We now assume that $\langle A \rangle$ is infinite and proceed in two steps:

(i) We show that $\langle A \rangle$ must contain elements of arbitrarily large lengths which contain no element of B as a segment.

(ii) Using (i), we obtain a contradiction.

(i) Let m be an arbitrary positive integer. Using the notation of Lemma 4, let

$$\begin{aligned} &k_0 = m, \quad k_1 = k(a_1b_1, k_0, A), \\ &k_2 = k(a_2b_2, k_0k_1, A), \ldots, \\ &k_p = k(a_pb_p, k_0 \cdots k_{p-1}, A). \end{aligned}$$

By finite induction, as in Lemma 2, it follows that if $x \in \langle A \rangle$, $|x| = k_0 \cdots k_p$, then x contains a segment R, $R \in \langle A \rangle$, $|R| = m$, such that no element of B is a segment of R.

(ii) First suppose that T is simple, so that C is empty. By (i), setting $m = 2$, there is $R \in \langle A \rangle$, $|R| = 2$, such that R contains no element of B as a segment, that is $R \notin B$. But $R \in B \cup C$, and C is empty. This case is finished.

Now suppose that T is 0-simple. By (i), for arbitrarily large m we have $R \in \langle A \rangle$, $|R| = m$, and no element of B is a segment of R. Then we can write $R = R'y$, where $y = \prod_{i=1}^{t}(x_iy_iz_iw_i)$, $R' \in \langle A \rangle$, $|R'| < 4$, $x_i, y_i, z_i, w_i \in A$, $|y_iz_i| = 2$, $1 \leq i \leq t$.

Then $y_i z_i \in C$, or $T((y_i z_i)\varphi)T = \{0\}$, therefore in particular $(x_i y_i z_i w_i)\varphi = 0$. Hence y is an element of a finitely generated (hence finite) subsemigroup of the locally finite semigroup $0\varphi^{-1}$, and so $|y|$ is bounded above. This contradicts the statement that $|R|$ is not bounded above. This finishes Lemma 4, and the proof of the main theorem is complete. □

5

In this section we are concerned with bands of locally finite semigroups. Suppose that a semigroup S is the disjoint union of certain subsemigroups S_α $(\alpha \in I)$, I an index set. Suppose further that for any pair α, β, of elements of I there is an element γ in I such that $S_\alpha S_\beta \subset S_\gamma$. Then S is a *bnad* of the semigroups S_α. Evidently I becomes an idempotent semigroup if we define $\alpha\beta = \gamma$ if and only if $S_\alpha S_\beta \subset S_\gamma$, and $\varphi : S \mapsto I$ is a homomorphism, where $x\varphi = \alpha$ if $x \in S_\alpha$. S is then called an *I-band* of the semigroups S_α. If each S_α is locally finite, then S is called simply an I-band of locally finite semigroups.

Several people ([4, 5, 7]) have shown idependently that an idempotent semigroup is locally finite. Thus from this and our main theorem follows the important result of Shevrin [8].

Theorem. *Any band of locally finite semigroups is locally finite.*

The author received in a personal communication from B. M. Schein an extremely short and direct proof of Shevrin's theorem which is outlined as follows: Let A be the two-element right zero semigroup, let B be the multiplicative semigroup $\{0,1\}$, let C be a right zero or left zero semigroup, let D be a rectangular band, and let E be a semilattice. It is shown that an X-band of locally finite semigroups is locally finite, where X is successively A, B, C, D, E, and Shevrin's theorem then follows since any band of locally finite semigroups is an E-band of D-bands of locally finite semigroups ([2, 3, 6]).

It is also true that any semigroup which is the union of disjoint locally finite groups is an E-band of D-bands of locally finite groups, and thus we have the next theorem.

Theorem. *A semigroup which is the union of locally finite groups is locally finite.*

From this follows the theorem of Green and Rees ([1, 4]) on the equivalence of Burnside's problem for groups with "Burnside's problem for semigroups":

Theorem. *The following two statements are equivalent:*

(1) Every group of exponent n is locally finite.

(2) Every semigroup satisfying the identity $x^{n+1} = x$ is locally finite.

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