A Semigroup Union of Disjoint Locally Finite Subsemigroups Which is Not Locally Finite

T. C. Brown

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Abstract

The semigroup S of the title is the free semigroup F on four generators factored by the congruence generated by the set of relations $\{w^2 = w^3 : w \in F\}$. The following lemma is proved by examining the elements of a given congruence class of F:

LEMMA. If $x, y \in S$ and $x^2 = y^2$, then either $xy = x^2$ or $yx = x^2$.

From the Lemma it then easily follows that the (disjoint) subsemigroups $\{y \in S : y^2 = x^2\}$ of S are locally finite.

This note answers in the negative a question raised by Shevrin in [2].

Theorem. *There exists a semigroup S with disjoint locally finite subsemigroups S_e such that $S = \bigcup S_e$ and S is not locally finite.*

Let F be the free semigroup with identity on four generators. Let $\sim$ denote the smallest congruence on F containing the set $\{(x^2, x^3) : x \in F\}$. That is, for $w, w' \in F$, $w \sim w'$ if and only if a finite sequence of “transitions", of either of the types $ab^2c \to ab^3c$ or $ab^3c \to ab^2c$, transforms w into w'.

The equivalence classes of F with respect to $\sim$ are taken as the elements of S, and multiplication in S is defined in the natural way.

There is given in [1] a sequence on four symbols in which no block of length k is immediately repeated, for any k. Thus the left initial segments of this sequence give elements of F containing no squares. Since no transition of the form $ab^2c \to ab^3c$ or $ab^3c \to ab^2c$ can be applied to an element of F contining no squares, the equivalence classes containing these elements consist of precisely one element each; thus the semigroup S is infinite, and hence not locally finite.

In what follows, the symbols $\alpha, \alpha_1, \alpha_2, \ldots$ refer to transformations (on elements of F) of the form

$$ab \to ayb, \quad \text{where } a \sim ay, \quad \text{and} \quad a, b, y \in F.$$

The symbols $\beta, \beta_1, \beta_2, \ldots$ refer to transformations of the type

$$axb \to ab, \quad \text{where } a \sim ax, \quad \text{and} \quad a, b, x \in F.$$

Note that $ab^2c \to ab^3c$ is an α, and $ab^3c \to ab^2c$ is a β.

Lemma 1. *If $w, w' \in F$, and $w\beta\alpha = w'$, then there are α_1, β_1 such that $w\alpha_1\beta_1 = w'$.*

Proof. Let

$$w = axb, \quad w\beta = ab, \quad \text{where } a \sim ax.$$

Let

$$w\beta = AB, \quad w\beta\alpha = AyB, \quad \text{where } A \sim Ay.$$

There are two cases:

(i) A is contained in a. That is,

$$a = Aa' \quad \text{and} \quad w' = wb\alpha = a\beta\alpha = Aa'b\alpha = Aya'b.$$

Here let

$$w\alpha_1 = axb\alpha_1 = Aa'xb\alpha_1 = Aya'xb.$$

Now since

$$Aya' \sim Aa' = a \sim ax = Aa'x \sim Aya'x,$$

we may let

$$w\alpha_1\beta_1 = Aya'xb\beta_1 = Aya'b = w'.$$

(ii) A is not contained in a. That is,

$$b = b_1b_2, \quad A = ab_1, \quad A \sim Ay,$$

and

$$w' = w\beta\alpha = ab\alpha = ab_1b_2\alpha = Ab_2\alpha = Ayb_2 = ab_1yb_2.$$

Since

$$axb_1 \sim ab_1 = A \sim Ay = ab_1y \sim axb_1y,$$

we may let

$$w\alpha_1 = axb\alpha_1 = axb_1b_2\alpha_1 = axb_1yb_2,$$

and

$$w\alpha_1\beta_1 = axb_1yb_2\beta_1 = ab_1yb_2 = w'. \qquad \square$$

Lemma 2. *If $w, w' \in F$, $w\gamma_1\gamma_2\cdots\gamma_m = w'$, where each γ_i is either an α or a β, then there are $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_k$ such that $w\alpha_1\cdots\alpha_n\beta_1\cdots\beta_k = w'$.*

Proof. This follows immediately from Lemma 1 by induction. □

Lemma 3. *The word $ab\alpha$ contains a left initial segment which is equivalent to a.*

Proof. Let $ab = AB$, $ab\alpha = AyB$, where $A \sim Ay$. Again there are two cases:

(i) A is contained in a. That is, $a = Aa'$, $ab\alpha = Aa'b\alpha = Aya'b$. Since $A \sim Ay$, the left initial segment Aya' of $ab\alpha$ is equivalent to a.

(ii) A is not contained in a. That is, $b = b_1b_2$, $A = ab_1$, $ab\alpha = ab_1b_2\alpha = ab_1yb_2$. Here, a itself is a left initial segment of $ab\alpha$, and is certainly equivalent to a. □

Lemma 4. *If $x, y \in F$ and $x^2 \sim y^2$, then either $y \sim xa$ for some $a \in F$, or $x \sim yb$ for some $b \in F$.*

Proof. By Lemma 2, thee are α_i and β_j such that $xx\alpha_1\cdots\alpha_m\beta_1\cdots\beta_n = yy$. Let $w = xx\alpha_1\cdots\alpha_m = yy\beta_n^{-1}\cdots\beta_1^{-1}$. By Lemma 3, w contains a left initial segment A equivalent to x. Similarly, since each β_i^{-1} is an α, w also contains a left initial segment B equivalent to y. Depending on which segment contains the other, either $B = Aa$ for some a, or $A = Bb$ for some b. In the first case, $y \sim B = Aa \sim xa$; in the second, $x \sim A = Bb \sim yb$. □

Lemma 5. *In this lemma, "=" will denote equality in S. Let e be an idempotent element of S: $e = e^2$. Let $S_e = \{x \in S : x^2 = e\}$. Then S_e is a locally finite subsemigroup of S.*

Proof. First, we note that $z \in S_e$ implies $ez = ze = e$. For $ez = z^2z = z^2 = e$, and similarly $ze = e$. Now let $x, y \in S_e$, that is, $x^2 = y^2 = e$. By Lemma 4, either $y = xa$ or $x = yb$. In the first case, we obtain

$$xy = x^2a = x^3a = x^2y = ey = e.$$

In the second case, we obtain similarly that $yx = e$. Thus $x, y \in S_e$ implies $xy = e$ or $yx = e$. In either case, $(xy)^2 = e$, that is, $xy \in S_e$. Thus S_e is a semigroup.

Now let $x_1, \ldots, x_n \in S_e$, and let $\langle x_1, \ldots, x_n \rangle$ denote the subsemigroup of S_e generated by $x_1, \ldots, x_n$. If $n = 1$, then $\langle x_1 \rangle$ is clearly finite; so suppose $n > 1$. Then every element of $\langle x_1, \ldots, x_n \rangle$ may be expressed as a product of not more than n of the x_i's. For any product z of more than n x_i's must contain some x_i twice: $z = ax_ibx_ic$, where $a, b, c \in S_e$. Since either $x_ib = e$ or $bx_i = e$, it follows that $x_ibx_i = e$ and $z = aec = ec = e = x_1x_1$. This shows that $\langle x_1, \ldots, x_n \rangle$ is finite, and hence that S_e is locally finite. □

The theorem follows immediately from Lemma 5, since clearly $e \neq e'$ implies that S_e and $S_{e'}$ are disjoint, and

$$S = \cup S_e.$$

References

[1] R. Dean, *A sequence without repeats on x, x^{-1}, y, y^{-1}*, Am. Math. Mon. **72** (1965), 383–385.

[2] L.N. Shevrin, *On locally finite semigroups*, Doklady Akad. Nauk SSSR **162** (1965), 770–773, (= Soviet Math. Dokl. 6 (1965) 769.).