On the Finiteness of Semigroups in Which $x^r = x$

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In this note we present a new proof of the following theorem (see [1]).

Theorem 1. *For each r, $r \geq 2$, every finitely generated semigroup in which $x^r = x$ holds identically is finite if and only if every finitely generated group of exponent $r-1$ is finite.*

Notation. Throughout, r is fixed and S_k denotes a semigroup on k generators in which $x^r = x$. The elements of S_k are regarded as equivalence classes of words in k symbols $X_1, \ldots, X_k$. Upper case letters will denote words, and lower case letters the elements of the semigroup S_k; thus if the word W represents the element w of S_k, we write $W \in w \in S_k$, and also say that W is a word of S_k. $W = W'$ signifies that W and W' are identical, while $W \sim W'$ signifies that W and W' are equivalent, i.e., represent the same element of S_k. If W is a word of S_k, we write $|W|$ for the *length* of W in the symbols X_i; e.g., $|X_1X_2X_3| = 3$. If each X_i appears in W, $1 \leq i \leq k$, we say that W is a *complete* word of S_k; any word W is *minimal* if $W \sim W'$ implies $|W| \leq |W'|$.

Remark 1. Clearly S_k is finite if and only if the lengths of minimal words of S_k are bounded from above.

Lemma 1. *$W \sim AXB$ implies $W \sim W(XW)^{r-1}$.*

Proof.
$$W(XW)^{r-1} \sim AXB(XAXB)^{r-1} \sim (AX)^{r-2}A[XAXB(XAXB)^{r-1}] \sim AXB \sim W.$$ □

Lemma 2. *Let T be a complete word of S_k. Then for any word X of S_k, $T \sim T(XT)^{r-1}$.*

Proof. By Lemma 1 we need only show that $T \sim AXB$ for some A, B. We show by induction on $|X|$ that $T \sim TXB$ for a suitable B. If $|X| = 0$, i.e., X is the empty word, we may take $B = X$. Now let $X = X_0X_i$, where $|X_0| = n$, and suppose $T \sim TX_0B_0$. Since T is complete we have $T = T'X_iT''$, and hence

$$\begin{aligned} T \sim T'(X_iT''X_0)B_0 &\sim (T'X_iT''X_0X_i)T''X_0(X_iT''X_0)^{r-2}B_0 \\ &= TX[T''X_0(X_iT''X_0)^{r-2}B_0]. \end{aligned}$$ □

Lemma 3. *Let $T \in t \in S_k$, where T is complete. Then the set tS_kt is a group.*

Proof. tS_kt is a semigroup with identity t^{r-1}, and the inverse of txt is $(txt)^{r-2}$. (For by Lemma 2 we have $t(xt)^{r-1} = t$; hence $(txt)^{r-1} = t(xtt)^{r-2}xt = t(xtt)^{r-1}t^{r-2} = t^{r-1}$.) □

Proof of Theorem. One direction is trivial. For the other, we use induction on k; S_1 is obviously finite, and we suppose that S_{k-1} is finite. By Remark 1, there is a number m such that if $|X| = m$, where X is a minimal word of S_k, then X is complete in the k symbols $X_1, \ldots, X_k$.

Let W be any minimal word of S_k; then W can be written in the form

$$W = W_1 W_2 \cdots W_N W', \tag{1}$$

where

$$|W_j| = m \quad (1 \leq i \leq N),\ |W'| < m.$$

Since W is minimal, each W_j is minimal and hence complete. Let $T \in t \in S_k$, where T is complete. Then for each j, $W_j \sim W_j(T^2 W_j)^{r-1}$; and making this substitution into (1) for each j, we obtain

$$W \sim W_1[T(TW_1T)^{r-2}(TW_1W_2T)(TW_2T)^{r-2} \cdots (TW_NT)^{r-2}T]W_NW'. \tag{2}$$

Now let G be the subgroup of tS_kt generated by all those elements with representatives of the form TXT, where $|X| \leq 2m$. Since G is finitely generated and of exponent $r-1$, G is finite by assumption; hence every element of G has a representative word whose length does not exceed some number M. Thus by (2), $W \sim W_1 Y W_N W'$, where $|Y| \leq M$, and by the minimality of W, $|W| \leq M + 3m$. Finally, since M did not depend on the particular choice of W, S_k is finite, and the proof is complete. □

References

[1] J.A. Green and D. Rees, *On semi-groups in which $x^r = x$*, Proc. Cambridge Philos. Soc. **48** (1952), 35–40.