A Simple Proof of Lerch's Formula

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1 Introduction

Using the convergents of the simple continued fraction for a positive irrational number $\alpha = [a_0, a_{1,2}, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$, it is possible to obtain an explicit formula for $\sum_{k=1}^{m} [k\alpha]$. This has been done several times. (See [4, 6, 7, 9–12]. These papers all deal with the asymptotic behaviour of the function $C_\alpha(m) = \sum_{k=1}^{m} (\{k\alpha\} - \frac{1}{2})$, where $\{x\}$ denotes the fractional part of x. Since $k\alpha = [k\alpha] + \{k\alpha\}$, $C_\alpha(m) = \sum_{k=1}^{m} (k\alpha - [k\alpha] - \frac{1}{2}) = \frac{m(m+1)\alpha}{2} - \sum_{k=1}^{m} [k\alpha] - \frac{m(m+1)}{4}$, any formula for $\sum_{k=1}^{m} [k\alpha]$ gives a formula for $C_\alpha(m)$ and conversely.)

The simplest formula for $\sum_{k=1}^{m} [k\alpha]$ is the following one, taken from [4].

When $\alpha < 1$, with $p_n/q_n = [0, a_1, a_2, \ldots, a_n]$ (p_n, q_n relatively prime), one has

$$\sum_{k=1}^{q_n} [k\alpha] = \frac{1}{2}(p_n q_n - q_n + p_n + (-1)^n).$$

Applying this to $\frac{1+\sqrt{5}}{2} = 1 + [0, 1, 1, \ldots]$ gives (after a little arithmetic) the identity

$$\sum_{k=1}^{F_n} \left[k \left(\frac{1+\sqrt{5}}{2} \right) \right] = \frac{1}{2}(F_n F_{n+1} + F_{n-1} - (-1)^n)$$

where $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$.

For general m, one first writes $m = z_t q_{t-1} + \cdots + z_2 q_1 + z_1 q_0$, where $0 \leq z_1 \leq a_1 - 1$; $0 \leq z_i \leq a_i$, $2 \leq i \leq t$; if $z_i = a_i$, then $z_{i-1} = 0, 2 \leq i \leq t$. (This is the so-called "Zeckendorff representation of m." To find it, subtract the largest possible q_j from m and repeat.) Then

$$\sum_{k=1}^{m} [k\alpha] = \frac{1}{2} \sum_{1 \leq i \leq t} z_i (z_i p_{i-1} q_{i-1} - q_{i-1} + p_{i-1} + (-1)^{i-1}) + \sum_{1 \leq i < j \leq t} z_i z_j q_{i-1} p_{j-1}$$

However, in the expression $\sum_{k=1}^{m} [k\alpha] + \sum_{k=1}^{[m\alpha]} [k\frac{1}{\alpha}]$, the complications arising in the two sums miraculously cancel each other out. M. Lerch [8] gave as a problem the identity $\sum_{k=1}^{m} [k\alpha] + \sum_{k=1}^{[m\alpha]} [k\frac{1}{\alpha}] = m[m\alpha]$. This identity is mentioned in [6] and [7].

The authors are not aware of any published proof of this formula, although it is certainly possible to give a (rather complicated) proof using a formula for $\sum_{k=1}^{m}[k\alpha]$. We give below a simple direct proof.

2 The Proof of Lerch's Formula

Let x be an irrational number, $x > 1$. Then the set $\{[nx] : n \geq 1\}$ is called a *Beatty sequence*, and is denoted by $x\mathbb{N}$. Our proof of Lerch's formula is based on the following interesting lemma, first popularized by Beatty [1]. (See [2], and the 12 references given in [5].) The papers [3] and [5] generalize this result to sequences of the form $\{[nx+z] : n \geq 1\}$.

Lemma 1. *Let $x > 1$ and $y > 1$ be two irrational numbers such that $\frac{1}{x}+\frac{1}{y} = 1$. Then the Beatty sequences $x\mathbb{N}$ and $y\mathbb{N}$ form a partition of $\mathbb{N} = \{1,2,\ldots\}$.*

Proof. For any $t \in \mathbb{N}$, define $A_t = x\mathbb{N} \cap (0,t)$, $B_t = y\mathbb{N} \cap (0,t)$.

Since $x > 1$, $y > 1$ are irrational, we obtain $\frac{t}{x} - 1 < |A_t| < \frac{t}{x}$ and $\frac{t}{y} - 1 < |B_t| < \frac{t}{y}$. Hence $t-2 < |A_t| + |B_t| < t$, so $|A_t| + |B_t| = t-1$, for each $t \geq 1$. Now by induction on t, it easily follows that the sets A_t and B_t form a partition of $\{1,2,\ldots,t-1\}$. □

As an immediate consequence we have:

Lemma 2. *Let $a > 0$ and $b > 0$ be two irrational numbers such that $\frac{1}{a}+\frac{1}{b} = 1$. Let n be a positive integer. Then*

$$\sum_{[ka]\leq n}[ka] + \sum_{[kb]\leq n}[kb] = \frac{1}{2}n(n+1)$$

Now we are almost ready to prove Lerch's formula. We will need the following facts about the floor function $\lfloor\cdot\rfloor$:

Lemma 3. *Let $\alpha > 0$, $x = 1+\alpha$, $y = 1+\frac{1}{\alpha}$, and let m,k be positive integers. Then*

(a) $\left\lfloor \lfloor m\alpha\rfloor \frac{1}{\alpha}\right\rfloor \leq m \leq \left\lfloor (m\alpha+1)\frac{1}{\alpha}\right\rfloor$

(b) $\lfloor ky\rfloor \leq \lfloor mx\rfloor$ *iff* $k \leq \lfloor m\alpha\rfloor$

Proof. Part (a) follows directly from the definition of the floor function. Obviously, $\lfloor ky\rfloor \leq \lfloor mx\rfloor$ is equivalent to $k + \left\lfloor k\frac{1}{\alpha}\right\rfloor \leq m + \lfloor m\alpha\rfloor$.

Now, if $k \leq \lfloor m\alpha\rfloor$ then, by part (a), $\left\lfloor k\frac{1}{\alpha}\right\rfloor \leq m$. Adding these two inequalities gives $k + \left\lfloor k\frac{1}{\alpha}\right\rfloor \leq m + \lfloor m\alpha\rfloor$.

On the other hand, if $k > \lfloor m\alpha\rfloor$, then $k \geq \lfloor m\alpha\rfloor + 1$, so by part (a), $\left\lfloor k\frac{1}{\alpha}\right\rfloor \geq m$. Adding the first and last of these inequalities gives $k + \left\lfloor k\frac{1}{\alpha}\right\rfloor > m + \lfloor m\alpha\rfloor$. □

Theorem 1. *(Lerch's formula) Let a be a positive irrational real number. Then for every positive integer m,*

$$\sum_{k=1}^{m}\lfloor k\alpha\rfloor + \sum_{k=1}^{\lfloor m\alpha\rfloor}\left\lfloor k\frac{1}{\alpha}\right\rfloor = m\lfloor m\alpha\rfloor$$

Proof. Let $x = 1+\alpha$, $y = 1+\frac{1}{\alpha}$. By Lemma 2 (with $n = \lfloor mx \rfloor$), we have

$$\begin{aligned}\sum_{\lfloor kx \rfloor \le \lfloor mx \rfloor} \lfloor kx \rfloor + \sum_{\lfloor ky \rfloor \le \lfloor mx \rfloor} \lfloor ky \rfloor &= \frac{1}{2}\lfloor mx \rfloor(\lfloor mx \rfloor + 1) \\ &= \frac{1}{2}(m+\lfloor m\alpha \rfloor)(m+\lfloor m\alpha \rfloor+1) \\ &= \frac{1}{2}m(m+1)+\frac{1}{2}\lfloor m\alpha \rfloor(\lfloor m\alpha \rfloor+1)+m\lfloor m\alpha \rfloor.\end{aligned}$$

On the other hand, by Lemma 3,

$$\begin{aligned}\sum_{\lfloor kx \rfloor \le \lfloor mx \rfloor} \lfloor kx \rfloor + \sum_{\lfloor ky \rfloor \le \lfloor mx \rfloor} \lfloor ky \rfloor &= \sum_{k=1}^{m} \lfloor kx \rfloor + \sum_{k=1}^{\lfloor m\alpha \rfloor} \lfloor ky \rfloor \\ &= \sum_{k=1}^{m} k + \lfloor k\alpha \rfloor + \sum_{k=1}^{\lfloor m\alpha \rfloor} \left(k + \left\lfloor k\frac{1}{\alpha} \right\rfloor\right) \\ &= \sum_{k=1}^{m} \lfloor k\alpha \rfloor + \sum_{k=1}^{\lfloor m\alpha \rfloor} \left\lfloor k\frac{1}{\alpha} \right\rfloor + \frac{1}{2}m(m+1) + \frac{1}{2}\lfloor m\alpha \rfloor(\lfloor m\alpha \rfloor+1)\end{aligned}$$

Lerch's formula follows. □

3 An Application

Let $\alpha = \frac{1+\sqrt{5}}{2}$, and let $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_{n+1} + F_n$, $n \ge 0$. When $n > 1$ and $i = 1,2,3$, we have the well-known identities

$$\lfloor F_n \alpha^i \rfloor = F_{n+i} - \frac{1}{2}((-1)^n + 1)$$

$$\left\lfloor F_{n+1}\frac{1}{\alpha^i} \right\rfloor = F_n - \frac{1}{2}\left((-1)^{n+1} + 1\right)$$

These, together with Lerch's formula, give

$$\sum_{k=1}^{F_n} \lfloor k\alpha \rfloor + \sum_{k=1}^{F_{n+1}} \left\lfloor k\frac{1}{\alpha} \right\rfloor = F_n F_{n+1}$$

$$\sum_{k=1}^{F_n} \lfloor k\alpha^2 \rfloor + \sum_{k=1}^{F_{n+2}} \left\lfloor k\frac{1}{\alpha^2} \right\rfloor = F_n F_{n+2}$$

$$\sum_{k=1}^{F_n} \lfloor k\alpha^3 \rfloor + \sum_{k=1}^{F_{n+3}} \left\lfloor k\frac{1}{\alpha^3} \right\rfloor = F_n F_{n+3}$$

Adding the first two equations, and equating the resulting right hand side with the right hand side of the third equation, gives the following pleasing identity, valid for $n > 1$:

$$\sum_{k=1}^{F_n} (\lfloor k\alpha \rfloor + \lfloor k\alpha^2 \rfloor) + \sum_{k=1}^{F_{n+1}} \left\lfloor k\frac{1}{\alpha} \right\rfloor + \sum_{k=1}^{F_{n+2}} \left\lfloor k\frac{1}{\alpha^2} \right\rfloor = \sum_{k=1}^{F_n} \lfloor k\alpha^3 \rfloor + \sum_{k=1}^{F_{n+3}} \left\lfloor k\frac{1}{\alpha^3} \right\rfloor$$

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