

Iterative Reweighted Minimization Methods for l_p Regularized Unconstrained Nonlinear Programming

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Abstract

In this paper we study general l_p regularized unconstrained minimization problems. In particular, we derive lower bounds for nonzero entries of the first- and second-order stationary points and hence also of local minimizers of the l_p minimization problems. We extend some existing iterative reweighted l_1 (IRL₁) and l_2 (IRL₂) minimization methods to solve these problems and propose new variants for them in which each sub-problem has a closed-form solution. Also, we provide a unified convergence analysis for these methods. In addition, we propose a novel Lipschitz continuous ϵ -approximation to $\|x\|_p^p$. Using this result, we develop new IRL₁ methods for the l_p minimization problems and show that any accumulation point of the sequence generated by these methods is a first-order stationary point, provided that the approximation parameter ϵ is below a computable threshold value. This is a remarkable result since all existing iterative reweighted minimization methods require that ϵ be dynamically updated and approach zero. Our computational results demonstrate that the new IRL₁ method and the new variants generally outperform the existing IRL₁ methods [20, 17].

Key words: l_p minimization, iterative reweighted l_1 minimization, iterative reweighted l_2 minimization

1 Introduction

Recently numerous optimization models and methods have been proposed for finding sparse solutions to a system or an optimization problem (e.g., see [30, 13, 8, 7, 23, 10, 9, 12, 31, 20,

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6, 2, 22, 32, 34, 26, 35]). In this paper we are interested in one of those models, namely, the l_p regularized unconstrained nonlinear programming model

$$\min_{x \in \mathfrak{R}^n} \{F(x) := f(x) + \lambda \|x\|_p^p\} \quad (1)$$

for some $\lambda > 0$ and $p \in (0, 1)$, where f is a smooth function with L_f -Lipschitz-continuous gradient in \mathfrak{R}^n , that is,

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L_f \|x - y\|_2, \quad \forall x, y \in \mathfrak{R}^n, \quad (2)$$

and f is bounded below in \mathfrak{R}^n . Here, $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for any $x \in \mathfrak{R}^n$. One can observe that as $p \downarrow 0$, problem (1) approaches the l_0 minimization problem

$$\min_{x \in \mathfrak{R}^n} f(x) + \lambda \|x\|_0, \quad (3)$$

which is an exact formulation of finding a sparse vector to minimize the function f . Some efficient numerical methods such as iterative hard thresholding [6] and penalty decomposition methods [26] have recently been proposed for solving (3). In addition, as $p \uparrow 1$, problem (1) approaches the l_1 minimization problem

$$\min_{x \in \mathfrak{R}^n} f(x) + \lambda \|x\|_1, \quad (4)$$

which is a widely used convex relaxation for (3). When f is a convex quadratic function, model (4) is shown to be extremely effective in finding a sparse vector to minimize f . A variety of efficient methods were proposed for solving (4) over last few years (e.g., see [31, 2, 22, 32, 34]). Since problem (1) is intermediate between problems (3) and (4), one can expect that it is also capable of seeking out a sparse vector to minimize f . As demonstrated by extensive computational studies in [10, 33], problem (1) can even produce a sparser solution than (4) does while both achieve similar values of f .

A great deal of effort was recently made by many researchers (e.g., see [10, 11, 12, 20, 33, 25, 16, 17, 19, 27, 14, 21, 29, 3, 15]) for studying problem (1) or its related problem

$$\min_{x \in \mathfrak{R}^n} \{\|x\|_p^p : Ax = b\}. \quad (5)$$

In particular, Chartrand [10], Chartrand and Staneva [11], Foucart and Lai [20], and Sun [29] established some sufficient conditions for recovering the sparsest solution to a undetermined linear system $Ax = b$ by model (5). Efficient iterative reweighted l_1 (IRL₁) and l_2 (IRL₂) minimization algorithms were also proposed for finding an approximate solution to (5) by Rao and Kreutz-Delgado [28], Chartrand and Yin [12], Foucart and Lai [20] and Daubechies et al. [19]. Though problem (5) is generally NP hard (see [14, 21]), it is shown in [19, Theorem 7.7(i)] that under some assumptions including a null space property on A and a posteriori check, the sequence generated by IRL₂ algorithm converges to the sparsest solution to the above linear system, which is also the global minimizer of (5). Mourad and Reilly [27] proposed a smooth

convex approximation approach to solving (5) in which $\|x\|_p^p$ is approximated by a smooth convex function at each iteration. In addition, Chen et al. [16] considered a special case of problem (1) with $f(x) = \frac{1}{2}\|Ax - b\|_2^2$, namely, the problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_p^p. \quad (6)$$

They derived lower bounds for nonzero entries of local minimizers of (6) and also proposed a hybrid orthogonal matching pursuit-smoothing gradient method for solving (6). Since $\|x\|_p^p$ is non-Lipschitz continuous, Chen and Zhou [17] recently considered the following approximation to (6):

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \lambda \sum_{i=1}^n (|x_i| + \epsilon)^p$$

for some small $\epsilon > 0$. And they also proposed an IRL₁ algorithm to solve this approximation problem. Recently, Lai and Wang [25] considered another approximation to (6), which is

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \lambda \sum_{i=1}^n (|x_i|^2 + \epsilon)^{p/2},$$

and proposed an IRL₂ algorithm for solving this approximation. Very recently, Bian and Chen [3] and Chen et al. [15] proposed a smoothing sequential quadratic programming (SQP) algorithm and a smoothing trust region Newton (TRN) method, respectively, for solving a class of nonsmooth nonconvex problems that include (1) as a special case. When applied to problem (1), their methods first approximate $\|x\|_p^p$ by a suitable smooth function and then apply an SQP or a TRN algorithm to solve the resulting approximation problem. Lately, Bian et al. [4] proposed first- and second-order interior point algorithms for solving a class of non-Lipschitz nonconvex minimization problems with *bounded* box constraints, which can be suitably applied to l_p regularized minimization problems over a *compact* box.

In this paper we consider general l_p regularized unconstrained optimization problem (1). In particular, we first derive lower bounds for nonzero entries of first- and second-order stationary points and hence also of local minimizers of (1). We then extend the aforementioned IRL₁ and IRL₂ methods [20, 19, 25, 17] to solve (1) and propose some new variants for them. We also provide a unified convergence analysis for these methods. Finally, we propose a novel Lipschitz continuous ϵ -approximation to $\|x\|_p^p$ and also propose a locally Lipschitz continuous function $F_\epsilon(x)$ to approximate $F(x)$. Subsequently, we develop IRL₁ minimization methods for solving the resulting approximation problem $\min_{x \in \mathbb{R}^n} F_\epsilon(x)$. We show that any accumulation point of the sequence generated by these methods is a first-order stationary point of problem (1), provided that ϵ is below a computable threshold value. This is a remarkable result since all existing iterative reweighted minimization methods for l_p minimization problems require that ϵ be dynamically updated and approach zero.

The outline of this paper is as follows. In Subsection 1.1 we introduce some notations that are used in the paper. In Section 2 we derive lower bounds for nonzero entries of stationary

points, and hence also of local minimizers of problem (1). We also propose a locally Lipschitz continuous function $F_\epsilon(x)$ to approximate $F(x)$ and study some properties of the approximation problem $\min_{x \in \mathfrak{R}^n} F_\epsilon(x)$. In Section 3 we extend the existing IRL₁ and IRL₂ minimization methods from problems (5) and (6) to general problems (1) and propose new variants for them. We also provide a unified convergence analysis for these methods. In Section 4 we propose new IRL₁ methods for solving (1) and establish their convergence. In Section 5 we conduct numerical experiments to compare the performance of the IRL₁ minimization methods and their variants that are studied in this paper for (1). Finally, in Section 6 we present some concluding remarks.

1.1 Notation

Given any $x \in \mathfrak{R}^n$ and a scalar τ , $|x|^\tau$ denotes an n -dimensional vector whose i th component is $|x_i|^\tau$. The set of all n -dimensional positive vectors is denoted by \mathfrak{R}_+^n . In addition, $x > 0$ means that $x \in \mathfrak{R}_+^n$ and $\text{Diag}(x)$ denotes an $n \times n$ diagonal matrix whose diagonal is formed by the vector x . Given an index set $\mathcal{B} \subseteq \{1, \dots, n\}$, $x_{\mathcal{B}}$ denotes the sub-vector of x indexed by \mathcal{B} . Similarly, $X_{\mathcal{B}\mathcal{B}}$ denotes the sub-matrix of X whose rows and columns are indexed by \mathcal{B} . In addition, if a matrix X is positive semidefinite, we write $X \succeq 0$. The sign operator is denoted by sgn , that is,

$$\text{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ [-1, 1] & \text{if } t = 0, \\ -1 & \text{otherwise.} \end{cases}$$

For any $\beta < 0$, we define $0^\beta = \infty$. Finally, we define

$$\underline{f} = \inf_{x \in \mathfrak{R}^n} f(x). \quad (7)$$

It follows from the early assumption on f that $-\infty < \underline{f} < \infty$.

2 Technical results

In this section we derive lower bounds for nonzero entries of stationary points and hence also of local minimizers of problem (1). We also propose a nonsmooth but locally Lipschitz continuous function $F_\epsilon(x)$ to approximate $F(x)$. Moreover, we show that when ϵ is below a computable threshold value, a certain stationary point of the corresponding approximation problem $\min_{x \in \mathfrak{R}^n} F_\epsilon(x)$ is also that of (1). This result plays a crucial role in developing new IRL₁ methods for solving (1) in Section 4.

2.1 Lower bounds for nonzero entries of stationary points of (1)

Chen et al. [15] recently studied optimality conditions for a class of non-Lipschitz optimization problems which include (1) as a special case. We first review some of these results in the context of problem (1). In particular, we will review the definition of first- and second-order

stationary points of problem (1) and state some necessary optimality conditions for (1). Then we derive lower bounds for nonzero entries of the stationary points and hence also of local minimizers of problem (1).

Definition 1 *Let x^* be a vector in \mathfrak{R}^n and $X^* = \text{Diag}(x^*)$. x^* is a first-order stationary point of (1) if*

$$X^* \nabla f(x^*) + \lambda p |x^*|^p = 0. \quad (8)$$

In addition, x^ is a second-order stationary point of (1) if*

$$(X^*)^T \nabla^2 f(x^*) X^* + \lambda p(p-1) \text{Diag}(|x^*|^p) \succeq 0. \quad (9)$$

The following result states that any local minimizer of (1) is a stationary point, whose proof can be found in [15].

Proposition 2.1 *Let x^* be a local minimizer of (1) and $X^* = \text{Diag}(x^*)$. The following statements hold:*

- (i) x^* is a first-order stationary point, that is, (8) holds at x^* .
- (ii) Further, if f is twice continuously differentiable in a neighborhood of x^* , then x^* is a second-order stationary point, that is, (9) holds at x^* .

Recently, Chen et al. derived in Theorems 3.1(i) and 3.3 of [16] some interesting lower bounds for the nonzero entries of local minimizers of a special case of problem (1) with $f(x) = \frac{1}{2} \|Ax - b\|^2$ for some $A \in \mathfrak{R}^{m \times n}$ and $b \in \mathfrak{R}^m$. We next establish similar lower bounds for the nonzero entries of stationary points and hence also of local minimizers of general problem (1).

Theorem 2.2 *Let x^* be a first-order stationary point of (1) satisfying $F(x^*) \leq F(x^0) + \epsilon$ for some $x^0 \in \mathfrak{R}^n$ and $\epsilon \geq 0$, $\mathcal{B} = \{i : x_i^* \neq 0\}$, L_f and \underline{f} be defined in (2) and (7), respectively. Then there holds:*

$$|x_i^*| \geq \left(\frac{\lambda p}{\sqrt{2L_f[F(x^0) + \epsilon - \underline{f}]}} \right)^{\frac{1}{1-p}}, \quad \forall i \in \mathcal{B}. \quad (10)$$

Proof. Since f has L_f -Lipschitz-continuous gradient in \mathfrak{R}^n , it is well-known that

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L_f}{2} \|y - x\|_2^2, \quad \forall x, y \in \mathfrak{R}^n.$$

Letting $x = x^*$ and $y = x^* - \nabla f(x^*)/L_f$, we obtain that

$$f(x^* - \nabla f(x^*)/L_f) \leq f(x^*) - \frac{1}{2L_f} \|\nabla f(x^*)\|_2^2. \quad (11)$$

Note that

$$f(x^* - \nabla f(x^*)/L_f) \geq \inf_{x \in \mathbb{R}^n} f(x) = \underline{f}, \quad f(x^*) \leq F(x^*) \leq F(x^0) + \epsilon.$$

Using these relations and (11), we have

$$\|\nabla f(x^*)\|_2 \leq \sqrt{2L_f[f(x^*) - f(x^* - \nabla f(x^*)/L_f)]} \leq \sqrt{2L_f[F(x^0) + \epsilon - \underline{f}]}. \quad (12)$$

Since x^* satisfies (8), we obtain that for every $i \in \mathcal{B}$,

$$|x_i^*| = \left(\frac{1}{\lambda p} \left| \frac{\partial f(x^*)}{\partial x_i} \right| \right)^{\frac{1}{p-1}} \geq \left(\frac{\|\nabla f(x^*)\|_2}{\lambda p} \right)^{\frac{1}{p-1}},$$

which together with (12) yields

$$|x_i^*| \geq \left(\frac{\lambda p}{\sqrt{2L_f[F(x^0) + \epsilon - \underline{f}]}} \right)^{\frac{1}{1-p}}, \quad \forall i \in \mathcal{B}.$$

■

Theorem 2.3 *Let x^* be a second-order stationary point of (1), $\mathcal{B} = \{i : x_i^* \neq 0\}$, and L_f be defined in (2). Suppose further that f is twice continuously differentiable in a neighborhood of x^* . Then there holds:*

$$|x_i^*| \geq \left(\frac{\lambda p(1-p)}{L_f} \right)^{\frac{1}{2-p}}, \quad \forall i \in \mathcal{B}, \quad (13)$$

Proof. It follows from (2) and the assumption that f is twice continuously differentiable in a neighborhood of x^* that $\|\nabla^2 f(x^*)\|_2 \leq L_f$. In addition, since x^* satisfies (9), we have

$$e_i^T [(X^*)^T \nabla^2 f(x^*) X^*] e_i + \lambda p(p-1) e_i^T \text{Diag}(|x^*|^p) e_i \geq 0,$$

where e_i is the i th coordinate vector. It then follows that for each $i \in \mathcal{B}$,

$$[\nabla^2 f(x^*)]_{ii} + \lambda p(p-1)|x_i^*|^{p-2} \geq 0,$$

which yields

$$|x_i^*| \geq \left(\frac{\lambda p(1-p)}{[\nabla^2 f(x^*)]_{ii}} \right)^{\frac{1}{2-p}} \geq \left(\frac{\lambda p(1-p)}{\|\nabla^2 f(x^*)\|_2} \right)^{\frac{1}{2-p}} \geq \left(\frac{\lambda p(1-p)}{L_f} \right)^{\frac{1}{2-p}}, \quad \forall i \in \mathcal{B}.$$

■

2.2 Locally Lipschitz continuous approximation to (1)

It is known that for $p \in (0, 1)$, the function $\|x\|_p^p$ is not locally Lipschitz continuous at some points in \mathfrak{R}^n . The non-Lipschitzness of $\|x\|_p^p$ brings a great deal of challenge for designing algorithms for solving problem (1) (see, for example, [16]). In this subsection we propose a nonsmooth but Lipschitz continuous ϵ -approximation to $\|x\|_p^p$ for every $\epsilon > 0$. As a consequence, we obtain a nonsmooth but locally Lipschitz continuous ϵ -approximation $F_\epsilon(x)$ to $F(x)$. Furthermore, we show that when ϵ is below a computable threshold value, a certain stationary point of the corresponding approximation problem $\min_{x \in \mathfrak{R}^n} F_\epsilon(x)$ is also that of (1).

Lemma 2.4 *Let $u > 0$ be arbitrarily given, and let q be such that*

$$\frac{1}{p} + \frac{1}{q} = 1. \quad (14)$$

Define

$$h_u(t) := \min_{0 \leq s \leq u} p \left(|t|s - \frac{s^q}{q} \right), \quad \forall t \in \mathfrak{R}. \quad (15)$$

Then the following statements hold:

(i) $0 \leq h_u(t) - |t|^p \leq u^q$ for every $t \in \mathfrak{R}$.

(ii) h_u is pu -Lipschitz continuous in $(-\infty, \infty)$, i.e.,

$$|h_u(t_1) - h_u(t_2)| \leq pu|t_1 - t_2|, \quad \forall t_1, t_2 \in \mathfrak{R}.$$

(iii) The Clarke subdifferential of h_u , denoted by ∂h_u , exists everywhere, and it is given by

$$\partial h_u(t) = p \min\{|t|^{\frac{1}{q-1}}, u\} \operatorname{sgn}(t). \quad (16)$$

Proof. (i) Let $g_t(s) = p(|t|s - s^q/q)$ for $s > 0$. Since $p \in (0, 1)$, we observe from (14) that $q < 0$. It then implies that $g_t(s) \rightarrow \infty$ as $s \downarrow 0$. This together with the continuity of g_t implies that $h_u(t)$ is well-defined for all $t \in \mathfrak{R}$. In addition, it is easy to show that $g_t(\cdot)$ is convex in $(0, \infty)$, and moreover, $\inf_{s>0} g_t(s) = |t|^p$. Hence, we have

$$h_u(t) = \min_{0 \leq s \leq u} g_t(s) \geq \inf_{s>0} g_t(s) = |t|^p, \quad \forall t \in \mathfrak{R}.$$

We next show that $h_u(t) - |t|^p \leq u^q$ by dividing its proof into two cases.

1) Assume that $|t| > u^{q-1}$. Then, the optimal value of (15) is achieved at $s^* = |t|^{\frac{1}{q-1}}$ and hence,

$$h_u(t) = p \left(|t|s^* - \frac{(s^*)^q}{q} \right) = |t|^p.$$

2) Assume that $|t| \leq u^{q-1}$. It can be shown that the optimal value of (15) is achieved at $s^* = u$. Using this result and the relation $|t| \leq u^{q-1}$, we obtain that

$$h_u(t) = p \left(|t|u - \frac{u^q}{q} \right) \leq p \left(u^{q-1}u - \frac{u^q}{q} \right) = u^q,$$

which implies that $h_u(t) - |t|^p \leq h_u(t) \leq u^q$.

Combining the above two cases, we conclude that statement (i) holds.

(ii) Let $\phi : [0, \infty) \rightarrow \Re$ be defined as follows:

$$\phi(t) = \begin{cases} t^p & \text{if } t > u^{q-1}, \\ p(tu - u^q/q) & \text{if } 0 \leq t \leq u^{q-1}. \end{cases}$$

It follows from (14) that $(q-1)(p-1) = 1$. Using this relation, one can show that

$$\phi'(t) = p \min\{t^{\frac{1}{q-1}}, u\}. \quad (17)$$

Hence, we can see that $0 \leq \phi'(t) \leq pu$ for every $t \in [0, \infty)$, which implies that ϕ is pu -Lipschitz continuous on $[0, \infty)$. In addition, one can observe from the proof of (i) that $h_u(t) = \phi(|t|)$ for all t . Further, by the triangle inequality, we can easily conclude that h_u is pu -Lipschitz continuous in $(-\infty, \infty)$.

(iii) Since h_u is Lipschitz continuous everywhere, it follows from Theorem 2.5.1 of [18] that

$$\partial h_u(t) = \text{cov} \left\{ \lim_{t_k \in D \rightarrow t} h'_u(t_k) \right\}, \quad (18)$$

where cov denotes convex hull and D is the set of points at which h_u is differentiable. Recall that $h_u(t) = \phi(|t|)$ for all t . Hence, $h'_u(t) = \phi'(|t|) \text{sgn}(t)$ for every $t \neq 0$. Using this relation, (17) and (18), we immediately see that statement (iii) holds. \blacksquare

Corollary 2.5 *Let $u > 0$ be arbitrarily given, and let $h(x) = \sum_{i=1}^n h_u(x_i)$ for every $x \in \Re^n$, where h_u is defined in (15). Then the following statements hold:*

(i) $0 \leq h(x) - \|x\|_p^p \leq nu^q$ for every $x \in \Re^n$.

(ii) h is $\sqrt{np}u$ -Lipschitz continuous in \Re^n , i.e.,

$$|h(x) - h(y)| \leq \sqrt{np}u \|x - y\|_2, \quad \forall x, y \in \Re^n.$$

We are now ready to propose a nonsmooth but locally Lipschitz continuous ϵ -approximation to $F(x)$.

Proposition 2.6 *Let $\epsilon > 0$ be arbitrarily given and q satisfy (14). Define*

$$F_\epsilon(x) := f(x) + \lambda \sum_{i=1}^n h_{u_\epsilon}(x_i), \quad (19)$$

where

$$h_{u_\epsilon}(t) := \min_{0 \leq s \leq u_\epsilon} p \left(|t|s - \frac{s^q}{q} \right), \quad u_\epsilon := \left(\frac{\epsilon}{\lambda n} \right)^{\frac{1}{q}}. \quad (20)$$

Then the following statements hold:

- (i) $0 \leq F_\epsilon(x) - F(x) \leq \epsilon$ for every $x \in \mathfrak{R}^n$.
- (ii) F_ϵ is locally Lipschitz continuous in \mathfrak{R}^n . Furthermore, if f is Lipschitz continuous, so is F_ϵ .

Proof. Using the definitions of F_ϵ and F , we have $F_\epsilon(x) - F(x) = \lambda(\sum_{i=1}^n h_{u_\epsilon}(x_i) - \|x\|_p^p)$, which, together with Corollary 2.5 (i) with $u = u_\epsilon$, implies that statement (i) holds. Since f is differentiable in \mathfrak{R}^n , it is known that f is locally Lipschitz continuous. In addition, we know from Corollary 2.5 (ii) that $\sum_{i=1}^n h_{u_\epsilon}(x_i)$ is Lipschitz continuous in \mathfrak{R}^n . These facts imply that statement (ii) holds. ■

From Proposition 2.6, we know that F_ϵ is a locally Lipschitz ϵ -approximation to the non-Lipschitz function F . It is very natural to find an approximate solution of (1) by solving the corresponding ϵ -approximation problem

$$\min_{x \in \mathfrak{R}^n} F_\epsilon(x), \quad (21)$$

where F_ϵ is defined in (19). Strikingly, we can show that when ϵ is below a computable threshold value, a certain stationary point of problem (21) is also that of (1).

Theorem 2.7 *Let $x^0 \in \mathfrak{R}^n$ be an arbitrary point, and let ϵ be such that*

$$0 < \epsilon < n\lambda \left[\frac{\sqrt{2L_f[F(x^0) + \epsilon - \underline{f}]}}{\lambda p} \right]^q, \quad (22)$$

where \underline{f} and q are defined in (7) and (14), respectively. Suppose that x^* is a first-order stationary point of (21) such that $F_\epsilon(x^*) \leq F_\epsilon(x^0)$. Then, x^* is also a first-order stationary point of (1), i.e., (8) holds at x^* . Moreover, the nonzero entries of x^* satisfy the first-order lower bound (10).

Proof. Let $\mathcal{B} = \{i : x_i^* \neq 0\}$. Since x^* is a first-order stationary point of (21), we have $0 \in \partial F_\epsilon(x^*)$. Hence, it follows that

$$\frac{\partial f(x^*)}{\partial x_i} + \lambda \partial h_{u_\epsilon}(x_i^*) = 0, \quad \forall i \in \mathcal{B}. \quad (23)$$

In addition, we notice that

$$f(x^*) \leq F(x^*) \leq F_\epsilon(x^*) \leq F_\epsilon(x^0) \leq F(x^0) + \epsilon. \quad (24)$$

Using this relation and similar arguments as for deriving (12), we see that (12) also holds for such x^* . It then follows from (23) and (12) that

$$|\partial h_{u_\epsilon}(x_i^*)| = \frac{1}{\lambda} \left| \frac{\partial f(x^*)}{\partial x_i} \right| \leq \frac{1}{\lambda} \|\nabla f(x^*)\|_2 \leq \frac{\sqrt{2L_f[F(x^0) + \epsilon - \underline{f}]}}{\lambda}, \quad \forall i \in \mathcal{B}. \quad (25)$$

We now claim that $|x_i^*| > u_\epsilon^{q-1}$ for all $i \in \mathcal{B}$, where u_ϵ is defined in (20). Suppose for contradiction that there exists some $i \in \mathcal{B}$ such that $0 < |x_i^*| \leq u_\epsilon^{q-1}$. It then follows from (16) that $|\partial h_{u_\epsilon}(x_i^*)| = pu_\epsilon$. Using this relation, (22) and the definition of u_ϵ , we obtain that

$$|\partial h_{u_\epsilon}(x_i^*)| = pu_\epsilon = p \left(\frac{\epsilon}{\lambda n} \right)^{1/q} > \frac{\sqrt{2L_f[F(x^0) + \epsilon - \underline{f}]}}{\lambda},$$

which contradicts (25). Therefore, $|x_i^*| > u_\epsilon^{q-1}$ for all $i \in \mathcal{B}$. Using this fact and (16), we see that $\partial h_{u_\epsilon}(x_i^*) = p|x_i^*|^{p-1} \text{sgn}(x_i^*)$ for every $i \in \mathcal{B}$. Substituting it into (23), we obtain that

$$\frac{\partial f(x^*)}{\partial x_i} + \lambda p|x_i^*|^{p-1} \text{sgn}(x_i^*) = 0, \quad \forall i \in \mathcal{B}.$$

Multiplying by x_i^* both sides of this equality, we see that (8) holds. In addition, recall from (24) that $F(x^*) \leq F(x^0) + \epsilon$. Using this relation and Theorem 2.2, we immediately see that the second part of this theorem also holds. \blacksquare

Remark. It is not hard to observe that Theorem 2.7 still holds if \underline{f} and L_f are replaced by a number below \underline{f} and a number above L_f in (22), respectively. For practical application, an upper bound on \overline{L}_f and a lower bound on \underline{f} thus suffice for estimating the parameter ϵ satisfying (22). These bounds can be easily found for some important regression problems such as the least squares and the logistic regressions (see Section 5). \blacksquare

Corollary 2.8 *Let $x^0 \in \mathfrak{R}^n$ be an arbitrary point, and let ϵ be such that (22) holds. Suppose that x^* is a local minimizer of (21) such that $F_\epsilon(x^*) \leq F_\epsilon(x^0)$. Then the following statements hold:*

- i) x^* is a first-order stationary point of (1), i.e., (8) holds at x^* . Moreover, the nonzero entries of x^* satisfy the first-order lower bound (10).
- ii) Suppose further that f is twice continuously differentiable in a neighborhood of x^* . Then, x^* is a second-order stationary point of (1), i.e., (9) holds at x^* . Moreover, the nonzero entries of x^* satisfy the second-order lower bound (13).

Proof. (i) Since x^* is a local minimizer of (21), we know that x^* is a stationary point of (21). Statement (i) then immediately follows from Theorem 2.7.

(ii) Let $\mathcal{B} = \{i : x_i^* \neq 0\}$. Since x^* is a local minimizer of (21), we observe that x^* is also a local minimizer of

$$\min_{x \in \mathbb{R}^n} \left\{ f(x) + \lambda \sum_{i \in \mathcal{B}} h_{u_\epsilon}(x_i) : x_i = 0, i \notin \mathcal{B} \right\}. \quad (26)$$

Notice that x^* is a first-order stationary point of (21). In addition, $F(x^*) \leq F(x^0) + \epsilon$ and ϵ satisfies (22). Using the same arguments as in the proof of Theorem 2.7, we have $|x_i^*| > u_\epsilon^{q-1}$ for all $i \in \mathcal{B}$. Recall from the proof of Lemma 2.4 (i) that $h_{u_\epsilon}(t) = |t|^p$ if $|t| > u_\epsilon^{q-1}$. Hence, $\sum_{i \in \mathcal{B}} h_{u_\epsilon}(x_i) = \sum_{i \in \mathcal{B}} |x_i|^p$ for all x in a neighborhood of x^* . This, together with the fact that x^* is a local minimizer of (26), implies that x^* is also a local minimizer of

$$\min_{x \in \mathbb{R}^n} \{f(x) + \lambda \|x_{\mathcal{B}}\|_p^p : x_i = 0, i \notin \mathcal{B}\}. \quad (27)$$

By the assumption, we observe that the objective function of (27) is twice continuously differentiable at x^* . The second-order optimality condition of (27) at x^* yields

$$\nabla^2 f(x^*)_{\mathcal{B}\mathcal{B}} + \lambda p(p-1) \text{Diag}(|x_{\mathcal{B}}^*|^{p-2}) \succeq 0,$$

which, together with the fact that $X^* = \text{Diag}(x^*)$ and $x_i^* = 0$ for $i \notin \mathcal{B}$, implies that (9) holds and hence x^* is a second-order stationary point of (1). The rest of the statement follows from Theorem 2.3. \blacksquare

3 A unified analysis for some existing iterative reweighted minimization methods

Recently two types of IRL₁ and IRL₂ methods have been proposed in the literature [20, 19, 25, 17] for solving problem (5) or (6). In this section we extend these methods to solve (1) and also propose a variant of them in which each subproblem has a closed-form solution. Moreover, we provide a unified convergence analysis for them.

3.1 The first type of IRL_α methods and its variant for (1)

In this subsection we consider the iterative reweighted minimization methods proposed in [25, 17] for solving problem (6), which apply an IRL₁ or IRL₂ method to solve a sequence of problems $\min_{x \in \mathbb{R}^n} Q_{1,\epsilon^k}(x)$ or $\min_{x \in \mathbb{R}^n} Q_{2,\epsilon^k}(x)$, where $\{\epsilon^k\}$ is a sequence of positive vectors approaching zero as $k \rightarrow \infty$ and

$$Q_{\alpha,\epsilon}(x) := \frac{1}{2} \|Ax - b\|_2^2 + \lambda \sum_{i=1}^n (|x_i|^\alpha + \epsilon_i)^{\frac{p}{\alpha}}. \quad (28)$$

In what follows, we extend the above methods to solve (1) and also propose a variant of them in which each subproblem has a closed-form solution. Moreover, we provide a unified convergence analysis for them. Our key observation is that problem

$$\min_{x \in \mathbb{R}^n} \{F_{\alpha, \epsilon}(x) := f(x) + \lambda \sum_{i=1}^n (|x_i|^\alpha + \epsilon_i)^{\frac{p}{\alpha}}\} \quad (29)$$

for $\alpha \geq 1$ and $\epsilon \in \mathbb{R}_+^n$ can be suitably solved by an iterative reweighted l_α (IRL $_\alpha$) method. Problem (1) can then be solved by applying the IRL $_\alpha$ method to a sequence of problems (29) with $\epsilon = \epsilon^k \in \mathbb{R}_+^n \rightarrow 0$ as $k \rightarrow \infty$.

We start by presenting an IRL $_\alpha$ method for solving problem (29) with $\alpha \geq 1$ and $\epsilon \in \mathbb{R}_+^n$, which becomes an IRL $_1$ (resp., IRL $_2$) method studied in [25, 17], respectively, when $\alpha = 1$ (resp., $\alpha = 2$) and $f(x) = \|Ax - b\|_2^2/2$.

Algorithm 1: An IRL $_\alpha$ minimization method for (29)

Let $\alpha \geq 1$ and $\epsilon \in \mathbb{R}_+^n$ be given. Choose an arbitrary $x^0 \in \mathbb{R}^n$. Set $k = 0$.

1) Solve the weighted l_α minimization problem

$$x^{k+1} \in \text{Arg min}_{x \in \mathbb{R}^n} \left\{ f(x) + \frac{\lambda p}{\alpha} \sum_{i=1}^n s_i^k |x_i|^\alpha \right\}, \quad (30)$$

where $s_i^k = (|x_i^k|^\alpha + \epsilon_i)^{\frac{p}{\alpha}-1}$ for all i .

2) Set $k \leftarrow k + 1$ and go to step 1).

end

We next show that the sequence $\{x^k\}$ generated above is bounded and moreover any accumulation point of $\{x^k\}$ is a first-order stationary point of (29).

Theorem 3.1 *Let the sequence $\{x^k\}$ be generated by the above IRL $_\alpha$ minimization method. There hold:*

- (i) *The sequence $\{x^k\}$ is bounded.*
- (ii) *Let x^* be any accumulation point of $\{x^k\}$. Then x^* is a first-order stationary point of (29).*

Proof. (i) Let q be such that

$$\frac{\alpha}{p} + \frac{1}{q} = 1. \quad (31)$$

It is not hard to show that for any $\delta > 0$,

$$(|t|^\alpha + \delta)^{\frac{p}{\alpha}} = \frac{p}{\alpha} \min_{s \geq 0} \left\{ (|t|^\alpha + \delta)s - \frac{s^q}{q} \right\}, \quad \forall t \in \mathbb{R}, \quad (32)$$

and moreover, the minimum is achieved at $s = (|t|^\alpha + \delta)^{\frac{1}{q-1}}$. Using this result, the definition of s^k , and (31), one can observe that for $k \geq 0$,

$$s^k = \min_{s \geq 0} G_{\alpha, \epsilon}(x^k, s), \quad x^{k+1} \in \text{Arg} \min_{x \in \mathfrak{R}^n} G_{\alpha, \epsilon}(x, s^k), \quad (33)$$

where $s^k = (s_1^k, \dots, s_n^k)^T$ and

$$G_{\alpha, \epsilon}(x, s) = f(x) + \frac{\lambda p}{\alpha} \sum_{i=1}^n \left[(|x_i|^\alpha + \epsilon_i) s_i - \frac{s_i^q}{q} \right]. \quad (34)$$

In addition, we see that $F_{\alpha, \epsilon}(x^k) = G_{\alpha, \epsilon}(x^k, s^k)$. It then follows that

$$F_{\alpha, \epsilon}(x^{k+1}) = G_{\alpha, \epsilon}(x^{k+1}, s^{k+1}) \leq G_{\alpha, \epsilon}(x^{k+1}, s^k) \leq G_{\alpha, \epsilon}(x^k, s^k) = F_{\alpha, \epsilon}(x^k). \quad (35)$$

Hence, $\{F_{\alpha, \epsilon}(x^k)\}$ is non-increasing. It follows that $F_{\alpha, \epsilon}(x^k) \leq F_{\alpha, \epsilon}(x^0)$ for all k . This together with (7), $\epsilon > 0$, and the definition of $F_{\alpha, \epsilon}$ implies that

$$\underline{f} + \lambda \sum_{i=1}^n |x_i^k|^p \leq \underline{f} + \lambda \sum_{i=1}^n (|x_i^k|^\alpha + \epsilon_i)^{\frac{p}{\alpha}} \leq f(x^k) + \lambda \sum_{i=1}^n (|x_i^k|^\alpha + \epsilon_i)^{\frac{p}{\alpha}} = F_{\alpha, \epsilon}(x^k) \leq F_{\alpha, \epsilon}(x^0).$$

It follows that $\|x^k\|_p^p \leq (F_{\alpha, \epsilon}(x^0) - \underline{f})/\lambda$ and hence $\{x^k\}$ is bounded.

(ii) Since x^* is an accumulation point of $\{x^k\}$, there exists a subsequence K such that $\{x^k\}_K \rightarrow x^*$. By the continuity of $F_{\alpha, \epsilon}$, we have $\{F_{\alpha, \epsilon}(x^k)\}_K \rightarrow F_{\alpha, \epsilon}(x^*)$, which together with the monotonicity of $F_{\alpha, \epsilon}(x^k)$ implies that $F_{\alpha, \epsilon}(x^k) \rightarrow F_{\alpha, \epsilon}(x^*)$. In addition, by the definition of s^k , we have $\{s^k\}_K \rightarrow s^*$, where $s^* = (s_1^*, \dots, s_n^*)^T$ with $s_i^* = (|x_i^*|^\alpha + \epsilon_i)^{\frac{p}{\alpha}-1}$ for all i . Also, we observe that $F_{\alpha, \epsilon}(x^*) = G_{\alpha, \epsilon}(x^*, s^*)$. Using (35) and $F_{\alpha, \epsilon}(x^k) \rightarrow F_{\alpha, \epsilon}(x^*)$, we see that $G_{\alpha, \epsilon}(x^{k+1}, s^k) \rightarrow F_{\alpha, \epsilon}(x^*) = G_{\alpha, \epsilon}(x^*, s^*)$. Further, it follows from (33) that $G_{\alpha, \epsilon}(x, s^k) \geq G_{\alpha, \epsilon}(x^{k+1}, s^k)$ for every $x \in \mathfrak{R}^n$. Upon taking limits on both sides of this inequality as $k \in K \rightarrow \infty$, we have $G_{\alpha, \epsilon}(x, s^*) \geq G_{\alpha, \epsilon}(x^*, s^*)$ for all $x \in \mathfrak{R}^n$, that is, $x^* \in \text{Arg} \min_{x \in \mathfrak{R}^n} G_{\alpha, \epsilon}(x, s^*)$, which, together with the first-order optimality condition and the definition of s^* , yields

$$0 \in \frac{\partial f(x^*)}{\partial x_i} + \lambda p (|x_i^*|^\alpha + \epsilon_i)^{\frac{p}{\alpha}-1} |x_i^*|^{\alpha-1} \text{sgn}(x_i^*), \quad \forall i. \quad (36)$$

Hence, x^* is a stationary point of (29). ■

The above IRL $_\alpha$ method needs to solve a sequence of reweighted l_α minimization subproblems (30) whose solution generally cannot be computed exactly. Therefore, the sequence $\{x^k\}$ usually can only be found inexactly. This may bring a great deal challenge to the practical implementation of this method due to the facts: 1) it is unknown how much inexactness on $\{x^k\}$ can be allowed to ensure the global convergence of the method; 2) it may not be cheap to find a good approximate solution to (30). Especially, when f is nonconvex, the subproblem (30) is also nonconvex and it is clearly hard to find x^k in this case. Thus, this method may

be practically inefficient or numerically unstable. We next propose a variant of this method in which each subproblem is much simpler and has a closed-form solution for the commonly used α 's such as $\alpha = 1$ or 2.

Algorithm 2: A variant of IRL $_{\alpha}$ minimization method for (29)

Let $\alpha \geq 1$, $0 < L_{\min} < L_{\max}$, $\tau > 1$ and $c > 0$ be given. Choose an arbitrary $x^0 \in \mathfrak{R}^n$ and set $k = 0$.

1) Choose $L_k^0 \in [L_{\min}, L_{\max}]$ arbitrarily. Set $L_k = L_k^0$.

1a) Solve the weighted l_{α} minimization problem

$$x^{k+1} \in \text{Arg min}_{x \in \mathfrak{R}^n} \left\{ f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L_k}{2} \|x - x^k\|_2^2 + \frac{\lambda p}{\alpha} \sum_{i=1}^n s_i^k |x_i|^{\alpha} \right\}, \quad (37)$$

where $s_i^k = (|x_i^k|^{\alpha} + \epsilon_i)^{\frac{p}{\alpha}-1}$ for all i .

1b) If

$$F_{\alpha, \epsilon}(x^k) - F_{\alpha, \epsilon}(x^{k+1}) \geq \frac{c}{2} \|x^{k+1} - x^k\|_2^2 \quad (38)$$

is satisfied, where $F_{\alpha, \epsilon}$ is given in (29), then go to step 2).

1c) Set $L_k \leftarrow \tau L_k$ and go to step 1a).

2) Set $k \leftarrow k + 1$ and go to step 1).

end

We first show that for each outer iteration, the number of its inner iterations is finite.

Theorem 3.2 *For each $k \geq 0$, the inner termination criterion (38) is satisfied after at most $\left\lceil \frac{\log(L_f + c) - \log(2L_{\min})}{\log \tau} + 2 \right\rceil$ inner iterations.*

Proof. Let $H(x)$ denote the objective function of (37). Notice that $H(\cdot)$ is strongly convex with modulus L_k due to $\alpha \geq 1$. By the first-order optimality condition of (37) at x^{k+1} , we have

$$H(x^k) \geq H(x^{k+1}) + \frac{L_k}{2} \|x^{k+1} - x^k\|_2^2,$$

which is equivalent to

$$f(x^k) + \frac{\lambda p}{\alpha} \sum_{i=1}^n s_i^k |x_i^k|^{\alpha} \geq f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{\lambda p}{\alpha} \sum_{i=1}^n s_i^k |x_i^{k+1}|^{\alpha} + L_k \|x^{k+1} - x^k\|_2^2.$$

Recall that ∇f is L_f -Lipschitz continuous. We then have

$$f(x^{k+1}) \leq f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L_f}{2} \|x^{k+1} - x^k\|_2^2.$$

Combining these two inequalities, we obtain that

$$f(x^k) + \frac{\lambda p}{\alpha} \sum_{i=1}^n s_i^k |x_i^k|^\alpha \geq f(x^{k+1}) + \frac{\lambda p}{\alpha} \sum_{i=1}^n s_i^k |x_i^{k+1}|^\alpha + (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2,$$

which together with (34) yields

$$G_{\alpha,\epsilon}(x^k, s^k) \geq G_{\alpha,\epsilon}(x^{k+1}, s^k) + (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2. \quad (39)$$

Recall that $F_{\alpha,\epsilon}(x^k) = G_{\alpha,\epsilon}(x^k, s^k)$. In addition, it follows from (32) that $F_{\alpha,\epsilon}(x) = \min_{s \geq 0} G_{\alpha,\epsilon}(x, s)$.

Using these two equalities and (39), we obtain that

$$\begin{aligned} F_{\alpha,\epsilon}(x^{k+1}) &= G_{\alpha,\epsilon}(x^{k+1}, s^{k+1}) \leq G_{\alpha,\epsilon}(x^{k+1}, s^k) \leq G_{\alpha,\epsilon}(x^k, s^k) - (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2 \\ &= F_{\alpha,\epsilon}(x^k) - (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2. \end{aligned}$$

Hence, (38) holds whenever $L_k \geq (L_f + c)/2$, which implies that L_k is updated only for a finite number of times. Let \bar{L}_k denote the final value of L_k at the k th outer iteration. It follows that $\bar{L}_k/\tau < (L_f + c)/2$, that is, $\bar{L}_k < \tau(L_f + c)/2$. Let n_k denote the number of inner iterations for the k th outer iteration. Then, we have

$$L_{\min} \tau^{n_k-1} \leq L_k^0 \tau^{n_k-1} = \bar{L}_k < \tau(L_f + c)/2.$$

Hence, $n_k \leq \left\lceil \frac{\log(L_f + c) - \log(2L_{\min})}{\log \tau} + 2 \right\rceil$ and the conclusion holds. \blacksquare

We next establish that the sequence $\{x^k\}$ generated above is bounded and moreover any accumulation point of $\{x^k\}$ is a first-order stationary point of problem (29).

Theorem 3.3 *Let $\{x^k\}$ be the sequence generated by the above variant of IRL $_{\alpha}$ method. There hold:*

- (i) *The sequence $\{x^k\}$ is bounded.*
- (ii) *Let x^* be any accumulation point of $\{x^k\}$. Then x^* is a first-order stationary point of (29).*

Proof. (i) It follows from (38) that $\{F_{\alpha,\epsilon}(x^k)\}$ is non-increasing and hence $F_{\alpha,\epsilon}(x^k) \leq F_{\alpha,\epsilon}(x^0)$ for all k . The rest of this proof is similar to that of Theorem 3.1 (i).

(ii) Since x^* is an accumulation point of $\{x^k\}$, there exists a subsequence K such that $\{x^k\}_K \rightarrow x^*$. By the continuity of $F_{\alpha,\epsilon}$, we have $\{F_{\alpha,\epsilon}(x^k)\}_K \rightarrow F_{\alpha,\epsilon}(x^*)$, which together with the monotonicity of $\{F_{\alpha,\epsilon}(x^k)\}$ implies that $F_{\alpha,\epsilon}(x^k) \rightarrow F_{\alpha,\epsilon}(x^*)$. Using this result and (38), we can conclude that $\|x^{k+1} - x^k\| \rightarrow 0$. Let \bar{L}_k denote the final value of L_k at the k th

outer iteration. From the proof of Theorem 3.2, we know that $\bar{L}_k \in [L_{\min}, \tau(L_f + c)/2]$. The first-order optimality condition of (37) with $L_k = \bar{L}_k$ yields

$$0 \in \frac{\partial f(x^k)}{\partial x_i} + \bar{L}_k(x_i^{k+1} - x_i^k) + \lambda p s_i^k |x_i^{k+1}|^{\alpha-1} \text{sgn}(x_i^{k+1}), \quad \forall i. \quad (40)$$

Noticing $\{s_i^k\}_K \rightarrow (|x_i^*| + \epsilon_i)^{\frac{p}{\alpha}-1}$ for all i and taking limits on both sides of (40) as $k \in K \rightarrow \infty$, we see that x^* satisfies (36) and hence x^* is a first-order stationary point of (29). \blacksquare

Corollary 3.4 *Let $\delta > 0$ be arbitrarily given, and let the sequence $\{x^k\}$ be generated by the above IRL $_{\alpha}$ method or its variant. Then, there exists some k such that*

$$\|X^k \nabla f(x^k) + \lambda p |X^k|^{\alpha} (|x^k|^{\alpha} + \epsilon)^{\frac{p}{\alpha}-1}\| \leq \delta,$$

where $X^k = \text{Diag}(x^k)$ and $|X^k|^{\alpha} = \text{Diag}(|x^k|^{\alpha})$.

Proof. As seen from Theorem 3.1 or 3.3, $\{x^k\}$ is bounded. Hence, $\{x^k\}$ has at least one accumulation point x^* . Moreover, it follows from these theorems that x^* satisfies (36). Multiplying by x_i^* both sides of (36), we have

$$x_i^* \frac{\partial f(x^*)}{\partial x_i} + \lambda p (|x_i^*|^{\alpha} + \epsilon_i)^{\frac{p}{\alpha}-1} |x_i^*|^{\alpha} = 0 \quad \forall i,$$

which, together with the continuity of $\nabla f(x)$ and $|x|^{\alpha}$, implies that the conclusion holds. \blacksquare

We are now ready to present the first type of IRL $_{\alpha}$ methods and its variant for solving problem (1) in which each subproblem is in the form of (29) and solved by the IRL $_{\alpha}$ or its variant described above. The IRL $_1$ and IRL $_2$ methods proposed in [25, 17] can be viewed as the special cases of the following general IRL $_{\alpha}$ method (but not its variant) with $f(x) = \|Ax - b\|_2^2/2$ and $\alpha = 1$ or 2 .

Algorithm 3: The first type of IRL $_{\alpha}$ minimization methods and its variant for (1)

Let $\alpha \geq 1$ be given, and $\{\delta_k\}$ and $\{\epsilon^k\}$ be a sequence of positive scalars and vectors, respectively. Choose an arbitrary $x^{0,0} \in \Re^n$ and set $k = 0$.

- 1) Apply the IRL $_{\alpha}$ method or its variant to problem (29) with $\epsilon = \epsilon^k$ starting at $x^{k,0}$ until finding x^k satisfying

$$\|X^k \nabla f(x^k) + \lambda p |X^k|^{\alpha} (|x^k|^{\alpha} + \epsilon^k)^{\frac{p}{\alpha}-1}\| \leq \delta_k, \quad (41)$$

where $X^k = \text{Diag}(x^k)$ and $|X^k|^{\alpha} = \text{Diag}(|x^k|^{\alpha})$.

- 2) Set $k \leftarrow k + 1$, $x^{k,0} \leftarrow x^{k-1}$ and go to step 1).

end

We next establish that the sequence $\{x^k\}$ generated by this IRL $_{\alpha}$ method or its variant is bounded and moreover any accumulation point of $\{x^k\}$ is a first-order stationary point of (1).

Theorem 3.5 Let $\{x^k\}$ be the sequence generated by the first type of IRL_α method or its variant. Suppose that $\{\epsilon^k\}$ is component-wise non-increasing, $\{\epsilon^k\} \rightarrow 0$ and $\{\delta_k\} \rightarrow 0$. There hold:

- (i) The sequence $\{x^k\}$ is bounded.
- (ii) Let x^* be any accumulation point of $\{x^k\}$. Then x^* is a first-order stationary point of (1), i.e., (8) holds at x^* .

Proof. (i) One can see from the proof of Theorem 3.1 (i) or 3.3 (i) that

$$F_{\alpha, \epsilon^{k+1}}(x^{k+1}) \leq F_{\alpha, \epsilon^{k+1}}(x^{k+1,0}) = F_{\alpha, \epsilon^{k+1}}(x^k), \quad (42)$$

where the equality is due to $x^{k+1,0} = x^k$. Since $\{\epsilon^k\}$ is non-increasing, we have $F_{\alpha, \epsilon^{k+1}}(x^k) \leq F_{\alpha, \epsilon^k}(x^k)$ for all k . This together with (42) implies that $F_{\alpha, \epsilon^{k+1}}(x^{k+1}) \leq F_{\alpha, \epsilon^k}(x^k)$ for all k , which yields $F_{\alpha, \epsilon^k}(x^k) \leq F_{\alpha, \epsilon^0}(x^0)$ for every k . The rest of the proof is similar to that of Theorem 3.1 (i).

(ii) Let $\mathcal{B} = \{i : x_i^* \neq 0\}$. It follows from (41) that

$$\left| x_i^k \frac{\partial f(x^k)}{\partial x_i} + \lambda p |x_i^k|^\alpha (|x_i^k|^\alpha + \epsilon_i^k)^{\frac{p}{\alpha} - 1} \right| \leq \delta_k \quad \forall i \in \mathcal{B}. \quad (43)$$

Since x^* is an accumulation point of $\{x^k\}$, there exists a subsequence K such that $\{x^k\}_K \rightarrow x^*$. Upon taking limits on both sides of (43) as $k \in K \rightarrow \infty$, we see that x^* satisfies (8) and it is a first-order stationary point of (1). \blacksquare

3.2 The second type of IRL_α methods and its variant for (1)

In this subsection we are interested in the IRL_1 and IRL_2 methods proposed in [20, 19] for solving problem (5). Given $\{\epsilon^k\} \subset \mathfrak{R}_+^n \rightarrow 0$ as $k \rightarrow \infty$, these methods solve a sequence of problems $\min_{x \in \mathfrak{R}^n} Q_{1, \epsilon^k}(x)$ or $\min_{x \in \mathfrak{R}^n} Q_{2, \epsilon^k}(x)$ extremely ‘‘roughly’’ by executing IRL_1 or IRL_2 method only one iteration for each ϵ^k , where $Q_{\alpha, \epsilon}$ is defined in (28).

We next extend the above methods to solve (1) and also propose a variant of them in which each subproblem has a closed-form solution. Moreover, we provide a unified convergence analysis for them. We start by presenting the second type of IRL_α methods for solving (1), which becomes the IRL_1 or IRL_2 method studied in [20, 19] when $\alpha = 1$ or 2, respectively.

Algorithm 4: The second type of IRL_α minimization method for (1)

Let $\alpha \geq 1$ be given and $\{\epsilon^k\} \subset \mathfrak{R}^n$ be a sequence of positive vectors. The rest of the algorithm is the same as Algorithm 1 except by replacing Step 1) by:

- 1) Solve problem (30) with $s_i^k = (|x_i^k|^\alpha + \epsilon_i^k)^{\frac{p}{\alpha} - 1}$ for all i to obtain x^{k+1} .

We next show that the sequence $\{x^k\}$ generated by this method is bounded and moreover any accumulation point of $\{x^k\}$ is a first-order stationary point of (1).

Theorem 3.6 Suppose that $\{\epsilon^k\}$ is a component-wise non-increasing sequence of positive vectors in \mathbb{R}^n and $\epsilon^k \rightarrow 0$ as $k \rightarrow \infty$. Let the sequence $\{x^k\}$ be generated by the second type of IRL $_{\alpha}$ method. There hold:

(i) The sequence $\{x^k\}$ is bounded.

(ii) Let x^* be any accumulation point of $\{x^k\}$. Then x^* is a first-order stationary point of (1), i.e., (8) holds at x^* .

Proof. (i) Let $G_{\alpha,\epsilon}(\cdot, \cdot)$ be defined in (34), and $s^k = (s_1^k, \dots, s_n^k)^T$, where s_i^k is defined above. One can observe that $G_{\alpha,\epsilon^k}(x^{k+1}, s^k) \leq G_{\alpha,\epsilon^k}(x^k, s^k)$. Also, by a similar argument as in the proof of Theorem 3.1, we have $G_{\alpha,\epsilon^{k+1}}(x^{k+1}, s^{k+1}) = \inf_{s \geq 0} G_{\alpha,\epsilon^{k+1}}(x^{k+1}, s)$. Hence, we obtain that $G_{\alpha,\epsilon^{k+1}}(x^{k+1}, s^{k+1}) \leq G_{\alpha,\epsilon^{k+1}}(x^{k+1}, s^k)$. Since $s^k > 0$ and $\{\epsilon^k\}$ is non-increasing, we observe that $G_{\alpha,\epsilon^{k+1}}(x^{k+1}, s^k) \leq G_{\alpha,\epsilon^k}(x^{k+1}, s^k)$. By these three inequalities, we have

$$G_{\alpha,\epsilon^{k+1}}(x^{k+1}, s^{k+1}) \leq G_{\alpha,\epsilon^{k+1}}(x^{k+1}, s^k) \leq G_{\alpha,\epsilon^k}(x^{k+1}, s^k) \leq G_{\alpha,\epsilon^k}(x^k, s^k), \quad \forall k \geq 0. \quad (44)$$

Hence, $\{G_{\alpha,\epsilon^k}(x^k, s^k)\}$ is non-increasing. By the definitions of s^k and $F_{\alpha,\epsilon}$, one can verify that

$$G_{\alpha,\epsilon^k}(x^k, s^k) = f(x^k) + \lambda \sum_{i=1}^n (|x_i^k|^\alpha + \epsilon_i^k)^{\frac{p}{\alpha}} = F_{\alpha,\epsilon^k}(x^k). \quad (45)$$

It follows that $\{F_{\alpha,\epsilon^k}(x^k)\}$ is non-increasing and hence $F_{\alpha,\epsilon^k}(x^k) \leq F_{\alpha,\epsilon^0}(x^0)$ for all k . The rest of the proof is similar to that of Theorem 3.5 (i).

(ii) Since x^* is an accumulation point of $\{x^k\}$, there exists a subsequence K such that $\{x^k\}_K \rightarrow x^*$. It then follows from $\epsilon^k \rightarrow 0$ and (45) that $\{G_{\alpha,\epsilon^k}(x^k, s^k)\}_K \rightarrow f(x^*) + \lambda \|x^*\|_p^p$. This together with the monotonicity of $\{G_{\alpha,\epsilon^k}(x^k, s^k)\}$ implies that $G_{\alpha,\epsilon^k}(x^k, s^k) \rightarrow f(x^*) + \lambda \|x^*\|_p^p$. Using this relation and (44), we further have

$$G_{\alpha,\epsilon^k}(x^{k+1}, s^k) \rightarrow f(x^*) + \lambda \|x^*\|_p^p. \quad (46)$$

Let $\mathcal{B} = \{i : x_i^* \neq 0\}$ and $\bar{\mathcal{B}}$ be its complement in $\{1, \dots, n\}$. We claim that

$$x^* \in \text{Arg min}_{x_{\bar{\mathcal{B}}}=0} \left\{ f(x) + \frac{\lambda p}{\alpha} \sum_{i \in \mathcal{B}} |x_i^*|^{p-\alpha} |x_i|^\alpha \right\}. \quad (47)$$

Indeed, using the definition of s^k , we see that $\{s_i^k\}_K \rightarrow |x_i^*|^{p-\alpha}$, $\forall i \in \mathcal{B}$. Due to $\epsilon^k > 0$, $x^k \geq 0$, $s^k > 0$ and $q < 0$, we further observe that

$$0 < \frac{p}{\alpha} \sum_{i \in \bar{\mathcal{B}}} \left[\epsilon_i^k s_i^k - \frac{(s_i^k)^q}{q} \right] \leq \frac{p}{\alpha} \sum_{i \in \bar{\mathcal{B}}} \left[(|x_i^k|^\alpha + \epsilon_i^k) s_i^k - \frac{(s_i^k)^q}{q} \right] = \sum_{i \in \bar{\mathcal{B}}} (|x_i^k|^\alpha + \epsilon_i^k)^{\frac{p}{\alpha}},$$

which, together with $\epsilon^k \rightarrow 0$ and $\{x_i^k\}_K \rightarrow 0$ for $i \in \bar{\mathcal{B}}$, implies that

$$\lim_{k \in K \rightarrow \infty} \sum_{i \in \bar{\mathcal{B}}} \left[\epsilon_i^k s_i^k - \frac{(s_i^k)^q}{q} \right] = 0. \quad (48)$$

In addition, by the definition of x^{k+1} , we know that $G_{\alpha, \epsilon^k}(x, s^k) \geq G_{\alpha, \epsilon^k}(x^{k+1}, s^k)$. Then for every $x \in \mathfrak{R}^n$ such that $x_{\bar{\mathcal{B}}} = 0$, we have

$$f(x) + \frac{\lambda p}{\alpha} \sum_{i \in \mathcal{B}} \left[(|x_i|^\alpha + \epsilon_i^k) s_i^k - \frac{(s_i^k)^q}{q} \right] + \frac{\lambda p}{\alpha} \sum_{i \in \bar{\mathcal{B}}} \left[\epsilon_i^k s_i^k - \frac{(s_i^k)^q}{q} \right] = G_{\alpha, \epsilon^k}(x, s^k) \geq G_{\alpha, \epsilon^k}(x^{k+1}, s^k).$$

Upon taking limits on both sides of this inequality as $k \in K \rightarrow \infty$, and using (46), (48) and the fact that $\{s_i^k\}_K \rightarrow |x_i^*|^{p-\alpha}$, $\forall i \in \mathcal{B}$, we obtain that

$$f(x) + \frac{\lambda p}{\alpha} \sum_{i \in \mathcal{B}} \left[|x_i|^\alpha |x_i^*|^{p-\alpha} - \frac{|x_i^*|^{q(p-\alpha)}}{q} \right] \geq f(x^*) + \lambda \|x^*\|_p^p$$

for all $x \in \mathfrak{R}^n$ such that $x_{\bar{\mathcal{B}}} = 0$. This inequality and (31) immediately yield (47). It then follows from (31) and the first-order optimality condition of (47) that x^* satisfies (8) and hence it is a stationary point of (1). \blacksquare

Notice that the subproblem of the above method generally cannot be solved exactly. For the similar reasons as mentioned earlier, this may bring a great deal of challenge to the implementation of this method. We next propose a variant of this method in which each subproblem is much simpler and has a closed-form solution for some commonly used α 's (e.g., $\alpha = 1$ or 2).

Algorithm 5: A variant of the second type of IRL $_\alpha$ minimization method for (1)

Let $\alpha \geq 1$ be given and $\{\epsilon^k\} \subset \mathfrak{R}^n$ be a sequence of positive vectors. The rest of the algorithm is the same as Algorithm 2 except by replacing Steps 1a) and 1b) by:

1a) Solve problem (37) with $s_i^k = (|x_i^k|^\alpha + \epsilon_i^k)^{\frac{p}{\alpha}-1}$ for all i to obtain x^{k+1} .

1b) If

$$F_{\alpha, \epsilon^k}(x^k) - F_{\alpha, \epsilon^{k+1}}(x^{k+1}) \geq \frac{c}{2} \|x^{k+1} - x^k\|_2^2 \quad (49)$$

is satisfied, then go to step 2).

We first show that for each outer iteration of the above method, the associated inner iterations terminate in a finite number of iterations.

Theorem 3.7 *For each $k \geq 0$, the inner termination criterion (49) is satisfied after at most $\left\lceil \frac{\log(L_f + c) - \log(2L_{\min})}{\log \tau} + 2 \right\rceil$ inner iterations.*

Proof. Let $G_{\alpha, \epsilon}(\cdot, \cdot)$ be defined in (34) and $s^k = (s_1^k, \dots, s_n^k)^T$, where s_i^k is defined above for all i . By a similar argument as in the proof of Theorem 3.2, one can show that (39) holds for all $k \geq 0$. In addition, similar as in the proof of Theorem 3.6, we can show that

$G_{\alpha, \epsilon^{k+1}}(x^{k+1}, s^{k+1}) \leq G_{\alpha, \epsilon^k}(x^{k+1}, s^k)$. Also, one can verify that $G_{\alpha, \epsilon^k}(x^k, s^k) = F_{\alpha, \epsilon^k}(x^k)$ for all k . Using these relations and (39), we obtain that

$$\begin{aligned} F_{\alpha, \epsilon^{k+1}}(x^{k+1}) &= G_{\alpha, \epsilon^{k+1}}(x^{k+1}, s^{k+1}) \leq G_{\alpha, \epsilon^k}(x^{k+1}, s^k) \leq G_{\alpha, \epsilon^k}(x^k, s^k) - (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2 \\ &= F_{\alpha, \epsilon^k}(x^k) - (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2. \end{aligned}$$

Hence, (49) holds whenever $L_k \geq (L_f + c)/2$. The rest of the proof is similar to that of Theorem 3.2. \blacksquare

We next show that the sequence $\{x^k\}$ generated by the variant of the second type of IRL $_{\alpha}$ method is bounded and moreover any accumulation point of $\{x^k\}$ is a first-order stationary point of (1).

Theorem 3.8 *Suppose that $\{\epsilon^k\}$ is a sequence of non-increasing positive vectors in \mathfrak{R}^n and $\epsilon^k \rightarrow 0$ as $k \rightarrow \infty$. Let the sequence $\{x^k\}$ be generated by the variant of the second type of IRL $_{\alpha}$ method. There hold:*

- (i) *The sequence $\{x^k\}$ is bounded.*
- (ii) *Let x^* be any accumulation point of $\{x^k\}$. Then x^* is a first-order stationary point of (1), i.e., (8) holds at x^* .*

Proof. (i) It follows from (49) that $\{F_{\alpha, \epsilon^k}(x^k)\}$ is non-increasing and hence $F_{\alpha, \epsilon^k}(x^k) \leq F_{\alpha, \epsilon^0}(x^0)$ for all k . The rest of this proof is similar to that of Theorem 3.5 (i).

(ii) Let \bar{L}_k denote the final value of L_k at the k th outer iteration. By similar arguments as in the proof of Theorem 3.3, we can show that $\bar{L}_k \in [L_{\min}, \tau(L_f + c)/2)$ and $\|x^{k+1} - x^k\| \rightarrow 0$. Let $\mathcal{B} = \{i : x_i^* \neq 0\}$. Since x^* is an accumulation point of $\{x^k\}$, there exists a subsequence K such that $\{x^k\}_K \rightarrow x^*$. By the definition of s_i^k , we see that $\lim_{k \in K \rightarrow \infty} s_i^k = |x_i^*|^{p-\alpha}$ for all $i \in \mathcal{B}$. The first-order optimality condition of (37) with $L_k = \bar{L}_k$ yields

$$\frac{\partial f(x^{k+1})}{\partial x_i} + \bar{L}_k(x_i^{k+1} - x_i^k) + \lambda p s_i^k |x_i^{k+1}|^{\alpha-1} \text{sgn}(x_i^{k+1}) = 0, \quad \forall i \in \mathcal{B}.$$

Upon taking limits on both sides of this equality as $k \in K \rightarrow \infty$, and using the relation $\lim_{k \in K \rightarrow \infty} s_i^k = |x_i^*|^{p-\alpha}$, one can see that x^* satisfies (8). \blacksquare

4 New iterative reweighted l_1 minimization for (1)

The IRL $_1$ and IRL $_2$ methods studied in Section 3 require that the parameter ϵ be dynamically adjusted and approach to zero. One natural question is whether an iterative reweighted minimization method can be proposed for (1) that shares a similar convergence with those methods but does not need to adjust ϵ . We will address this question by proposing a new IRL $_1$ method and its variant.

As shown in Subsection 2.2, problem (21) has a locally Lipschitz continuous objective function and it is an ϵ -approximation to (1). Moreover, when ϵ is below a computable threshold value, a certain stationary point of (21) is also that of (1). Therefore, it is natural to find an approximate solution of problem (1) by solving (21). In this section we propose new IRL₁ methods for solving (1), which can be viewed as the IRL₁ methods directly applied to problem (21). The novelty of these methods is in that the parameter ϵ is chosen only once and then fixed throughout all iterations. Remarkably, we are able to establish that any accumulation point of the sequence generated by these methods is a first-order stationary point of (1).

Algorithm 6: A new IRL₁ minimization method for (1)

Let q be defined in (14). Choose an arbitrary $x^0 \in \mathfrak{R}^n$ and ϵ such that (22) holds. Set $k = 0$.

1) Solve the weighted l_1 minimization problem

$$x^{k+1} \in \text{Arg} \min_{x \in \mathfrak{R}^n} \left\{ f(x) + \lambda p \sum_{i=1}^n s_i^k |x_i| \right\}, \quad (50)$$

where $s_i^k = \min \left\{ \left(\frac{\epsilon}{\lambda n} \right)^{\frac{1}{q}}, |x_i^k|^{\frac{1}{q-1}} \right\}$ for all i .

2) Set $k \leftarrow k + 1$ and go to step 1).

end

We next show that the sequence $\{x^k\}$ generated by this method is bounded and moreover any accumulation point of $\{x^k\}$ is a first-order stationary point of (1).

Theorem 4.1 *Let the sequence $\{x^k\}$ be generated by the new IRL₁ method. Assume that ϵ satisfies (22). There hold:*

- (i) *The sequence $\{x^k\}$ is bounded.*
- (ii) *Let x^* be any accumulation point of $\{x^k\}$. Then x^* is a first-order stationary point of (1), i.e., (8) holds at x^* . Moreover, the nonzero entries of x^* satisfy the first-order bound (10).*

Proof. (i) Let $u_\epsilon = \left(\frac{\epsilon}{\lambda n} \right)^{1/q}$, $s^k = (s_1^k, \dots, s_n^k)^T$, and

$$G(x, s) = f(x) + \lambda p \sum_{i=1}^n \left[|x_i| s_i - \frac{s_i^q}{q} \right].$$

By the definition of $\{s^k\}$, one can observe that for $k \geq 0$,

$$s^k = \arg \min_{0 \leq s \leq u_\epsilon} G(x^k, s), \quad x^{k+1} \in \text{Arg} \min_{x \in \mathfrak{R}^n} G(x, s^k). \quad (51)$$

In addition, we observe that $F_\epsilon(x) = \min_{0 \leq s \leq u_\epsilon} G(x, s)$ and $F_\epsilon(x^k) = G(x^k, s^k)$ for all k , where F_ϵ is defined in (19). It then follows that

$$F_\epsilon(x^{k+1}) = G(x^{k+1}, s^{k+1}) \leq G(x^{k+1}, s^k) \leq G(x^k, s^k) = F_\epsilon(x^k). \quad (52)$$

Hence, $\{F_\epsilon(x^k)\}$ is non-increasing and $F_\epsilon(x^k) \leq F_\epsilon(x^0)$ for all k . This together with Proposition 2.6 (i) implies that $F(x^k) \leq F_\epsilon(x^0)$. Using this relation, (1) and (7), we see that $\|x^k\|_p^p \leq (F_\epsilon(x^0) - f)/\lambda$ and hence $\{x^k\}$ is bounded.

(ii) Since x^* is an accumulation point of $\{x^k\}$, there exists a subsequence K such that $\{x^k\}_K \rightarrow x^*$. By the continuity of F_ϵ , we have $\{F_\epsilon(x^k)\}_K \rightarrow F_\epsilon(x^*)$, which together with the monotonicity of $\{F_\epsilon(x^k)\}$ implies that $F_\epsilon(x^k) \rightarrow F_\epsilon(x^*)$. Let $s_i^* = \min\{u_\epsilon, |x_i^*|^{\frac{1}{q-1}}\}$ for all i . We then observe that $\{s^k\}_K \rightarrow s^*$ and $F_\epsilon(x^*) = G(x^*, s^*)$. Using (52) and $F_\epsilon(x^k) \rightarrow F_\epsilon(x^*)$, we see that $G(x^{k+1}, s^k) \rightarrow F_\epsilon(x^*) = G(x^*, s^*)$. Also, it follows from (51) that $G(x, s^k) \geq G(x^{k+1}, s^k)$ for all $x \in \mathfrak{R}^n$. Taking limits on both sides of this inequality as $k \in K \rightarrow \infty$, we have $G(x, s^*) \geq G(x^*, s^*)$ for all $x \in \mathfrak{R}^n$. Hence, we have $x^* \in \text{Arg min}_{x \in \mathfrak{R}^n} G(x, s^*)$, whose first-order optimality condition yields

$$0 \in \frac{\partial f(x^*)}{\partial x_i} + \lambda p s_i^* \text{sgn}(x_i^*), \quad \forall i. \quad (53)$$

Recall that $s_i^* = \min\{u_\epsilon, |x_i^*|^{\frac{1}{q-1}}\}$. Substituting it into (53) and using (16), we obtain that

$$0 \in \frac{\partial f(x^*)}{\partial x_i} + \lambda \partial h_{u_\epsilon}(x_i^*), \quad \forall i.$$

It then follows from (19) that x^* is a first-order stationary point of F_ϵ . In addition, by the monotonicity of $\{F_\epsilon(x^k)\}$ and $F_\epsilon(x^k) \rightarrow F_\epsilon(x^*)$, we know that $F_\epsilon(x^*) \leq F_\epsilon(x^0)$. Using these results and Theorem 2.7, we conclude that x^* is a first-order stationary point of (1). The rest of conclusion immediately follows from Theorem 2.2. \blacksquare

The subproblem (50) of the above IRL₁ method generally does not have a closed-form solution. Therefore, it requires some numerical method to find an approximate solution instead. Due to the similar reasons as mentioned in Section 3, this method may be practically inefficient or numerically unstable. We next propose a variant of this method in which each subproblem has a *closed-form* solution.

Algorithm 7: A variant of new IRL₁ minimization method for (1)

Let $0 < L_{\min} < L_{\max}$, $\tau > 1$ and $c > 0$ be given. Let q be defined in (14). Choose an arbitrary x^0 and ϵ such that (22) holds. Set $k = 0$.

- 1) Choose $L_k^0 \in [L_{\min}, L_{\max}]$ arbitrarily. Set $L_k = L_k^0$.

1a) Solve the weighted l_1 minimization problem

$$x^{k+1} \in \text{Arg min}_{x \in \mathfrak{R}^n} \left\{ f(x^k) + \nabla f(x^k)^T (x - x^k) + \frac{L_k}{2} \|x - x^k\|_2^2 + \lambda p \sum_{i=1}^n s_i^k |x_i| \right\}, \quad (54)$$

where $s_i^k = \min \left\{ \left(\frac{\epsilon}{\lambda n} \right)^{\frac{1}{q}}, |x_i^k|^{\frac{1}{q-1}} \right\}$ for all i .

1b) If

$$F_\epsilon(x^k) - F_\epsilon(x^{k+1}) \geq \frac{c}{2} \|x^{k+1} - x^k\|_2^2 \quad (55)$$

is satisfied, where F_ϵ is defined in (19), then go to step 2).

1c) Set $L_k \leftarrow \tau L_k$ and go to step 1a).

2) Set $k \leftarrow k + 1$ and go to step 1).

end

We first show that for each outer iteration, the number of its inner iterations is finite.

Theorem 4.2 *For each $k \geq 0$, the inner termination criterion (55) is satisfied after at most $\left\lceil \frac{\log(L_f + c) - \log(2L_{\min})}{\log \tau} + 2 \right\rceil$ inner iterations.*

Proof. By a similar argument as for proving (39), one can show that for all $k \geq 0$,

$$G(x^k, s^k) \geq G(x^{k+1}, s^k) + (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2. \quad (56)$$

Recall that $F_\epsilon(x) = \min_{0 \leq s \leq u_\epsilon} G(x, s)$ and $F_\epsilon(x^k) = G(x^k, s^k)$, where $u_\epsilon = (\frac{\epsilon}{\lambda n})^{1/q}$. Using these two relations and (56), we obtain that

$$\begin{aligned} F_\epsilon(x^{k+1}) &= G(x^{k+1}, s^{k+1}) \leq G(x^{k+1}, s^k) \leq G(x^k, s^k) - (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2 \\ &= F_\epsilon(x^k) - (L_k - \frac{L_f}{2}) \|x^{k+1} - x^k\|_2^2. \end{aligned}$$

Hence, (55) holds whenever $L_k \geq (L_f + c)/2$. The rest of the proof is similar to that of Theorem 3.2. \blacksquare

We next establish that the sequence $\{x^k\}$ generated by the above variant of new IRL₁ method is bounded and moreover any accumulation point of $\{x^k\}$ is a first-order stationary point of (1).

Theorem 4.3 *Let the sequence $\{x^k\}$ be generated by the above variant of new IRL₁ method. Assume that ϵ satisfies (22). There hold:*

(i) *The sequence $\{x^k\}$ is bounded.*

(ii) Let x^* be any accumulation point of $\{x^k\}$. Then x^* is a first-order stationary point of (1), i.e., (8) holds at x^* . Moreover, the nonzero entries of x^* satisfy the first-order bound (10).

Proof. (i) It follows from (55) that $\{F_\epsilon(x^k)\}$ is non-increasing. The rest of the proof is similar to that of Theorem 4.1 (i).

(ii) Let \bar{L}_k denote the final value of L_k at the k th outer iteration. By similar arguments as in the proof of Theorem 3.3, we can show that $\bar{L}_k \in [L_{\min}, \tau(L_f + c)/2)$ and $\|x^{k+1} - x^k\| \rightarrow 0$. Let $\mathcal{B} = \{i : x_i^* \neq 0\}$. Since x^* is an accumulation point of $\{x^k\}$, there exists a subsequence K such that $\{x^k\}_K \rightarrow x^*$. The first-order optimality condition of (54) with $L_k = \bar{L}_k$ yields

$$0 \in \frac{\partial f(x^k)}{\partial x_i} + \bar{L}_k(x_i^{k+1} - x_i^k) + \lambda p s_i^k \operatorname{sgn}(x_i^{k+1}) = 0, \quad \forall i.$$

Upon taking limits on both sides of this equality as $k \in K \rightarrow \infty$, we see that (53) holds with $s_i^* = \min\{(\frac{\epsilon}{\lambda n})^{1/q}, |x_i^*|^{\frac{1}{q-1}}\}$ for all i . The rest of the proof is similar to that of Theorem 4.1. ■

5 Computational results

In this section we conduct numerical experiment to compare the performance of the IRL₁ methods (Algorithms 3, 4, 6) and their variants (Algorithms 3, 5, 7) studied in Subsections 3.1 and 3.2 and Section 4. In particular, we apply these methods to problem (1) with f being chosen as a least squares loss and a logistic loss, respectively, on randomly generated data. For convenience of presentation, we name these IRL₁ methods as IRL₁-1, IRL₁-2 and IRL₁-3, and their variants as IRL₁-1-v, IRL₁-2-v and IRL₁-3-v, respectively. All codes are written in MATLAB and all computations are performed on a MacBook Pro running with Mac OS X Lion 10.7.4 and 4GB memory.

The same initial point x^0 is used for all methods. In particular, we choose x^0 to be a solution of (4), which can be computed by a variety of methods (e.g., [31, 2, 22, 32, 34]). And all methods terminate according to the following criterion

$$\|\operatorname{Diag}(x)\nabla f(x) + \lambda p|x|^p\|_\infty \leq 10^{-6}.$$

For the method IRL₁-1 (Algorithms 3), we set $\delta_k = 0.1^k$ and $\epsilon^k = 0.1^k e$, where e is the all-ones vector. In this method, for each pair of (δ_k, ϵ^k) , Algorithm 1 is called to find a x^k satisfying (41), whose subproblem (30) with $\alpha = 1$ is solved by the spectral projected gradient (SPG) method [32] with the termination criterion

$$\|\operatorname{Diag}(x)\nabla f(x) + \lambda p \operatorname{Diag}(s^k)|x|\|_\infty \leq \max\{0.995^i, 0.1\delta_k\}, \quad (57)$$

where s^k is given in Step 1) of Algorithm 1 and i denotes the number of subproblem (30) solved so far in the current call of Algorithm 1. We set $\epsilon^k = 0.5^k e$ for the method IRL₁-2 (Algorithm 4). In addition, each subproblem of IRL₁-2 and IRL₁-3 (Algorithm 6) is solved by the SPG method [32] with a similar termination criterion as (57).

For the method IRL₁-2-v (Algorithm 5), we choose $\epsilon^k = 0.5^k e$. Also, for this method and IRL₁-3-v (Algorithm 7), we set $L_{\min} = 10^{-8}$, $L_{\max} = 10^8$, $c = 10^{-4}$, $\tau = 1.1$, and $L_0^0 = 1$. And we update L_k^0 by the same strategy as used in [1, 5, 32], that is,

$$L_k^0 = \max \left\{ L_{\min}, \min \left\{ L_{\max}, \frac{\Delta x^T \Delta g}{\|\Delta x\|^2} \right\} \right\},$$

where $\Delta x = x^k - x^{k-1}$ and $\Delta g = \nabla f(x^k) - \nabla f(x^{k-1})$. In addition, for the method IRL₁-1-v (Algorithm 3), we choose $\delta_k = 0.1^k$ and $\epsilon^k = 0.1^k e$. In this method, for each pair of (δ_k, ϵ^k) , Algorithm 2 is employed to find a x^k satisfying (41) with $\alpha = 1$. And the parameters for Algorithm 2 are set to be the same as those for IRL₁-2-v and IRL₁-3-v mentioned above.

In the first experiment, we compare the performance of the above methods for solving problem (1) with $\lambda = 3 \times 10^{-3}$ and

$$f(x) = \frac{1}{2} \|Ax - b\|_2^2 \quad (\text{least squares loss}).$$

It is easy to see that $\underline{f} \geq 0$ and the Lipschitz constant of ∇f is $L_f = \|A\|^2$. As remarked in the end of Section 2, for IRL₁-3 and IRL₁-3-v, ϵ can be chosen as the one satisfying (22) but within 10^{-6} to the supremum of all ϵ 's satisfying (22) with \underline{f} being replaced by 0.

We randomly generate matrix A and vector b with entries randomly chosen from standard uniform distribution. The results of these methods with $p = 0.1$ and 0.5 on these data are presented in Tables 1-4, respectively. In detail, the parameters m and n of each instance are listed in the first two columns, respectively. The objective function value of problem (6) for these methods is given in columns three to five, and CPU times (in seconds) are given in the last three columns, respectively. We shall mention that the CPU time reported here does not include the time for obtaining initial point x^0 . We can observe that: 1) all methods produce similar objective function values; 2) the new IRL₁ method (i.e., IRL₁-3) is generally faster than the other two IRL₁ methods, namely, IRL₁-1 and IRL₁-2; 3) the variant of the new IRL₁ method (i.e., IRL₁-3-v) is generally faster than IRL₁-1-v and IRL₁-2-v that are the variants of the other two IRL₁ methods; 4) the method IRL₁- i -v, namely, the variant of IRL₁- i , substantially outperforms the IRL₁- i method in terms of CPU time for $i = 1, 2, 3$; and 5) the variant of the new IRL₁ method (namely, IRL₁-3-v) is generally faster than all other methods.

In the second experiment, we compare the performance of the above methods for solving problem (1) with $\lambda = 3 \times 10^{-3}$ and

$$f(x) = \sum_{i=1}^m \log(1 + \exp(-b_i (a^i)^T x)) \quad (\text{logistic loss}).$$

It can be verified that $\underline{f} \geq 0$ and the Lipschitz constant of ∇f is $L_f = \|\tilde{A}\|^2$, where

$$\tilde{A} = [b_1 a^1, \dots, b_m a^m].$$

Similar as above, for IRL₁-3 and IRL₁-3-v, ϵ is chosen as the one satisfying (22) but within 10^{-6} to the supremum of all ϵ 's satisfying (22) with \underline{f} being replaced by 0.

Table 1: Comparison of three IRL_1 methods for least squares loss with $p = 0.1$

Problem		Objective Value			CPU Time		
m	n	IRL_1-1	IRL_1-2	IRL_1-3	IRL_1-1	IRL_1-2	IRL_1-3
100	500	0.174	0.177	0.175	7.7	4.7	3.7
200	1000	0.338	0.338	0.338	10.3	11.6	8.9
300	1500	0.465	0.465	0.465	30.6	31.1	30.3
400	2000	0.616	0.616	0.616	94.0	83.5	78.8
500	2500	0.776	0.776	0.776	215.4	218.9	198.7
600	3000	0.923	0.923	0.923	541.1	543.4	499.8
700	3500	1.076	1.069	1.051	353.3	510.4	816.5
800	4000	1.213	1.213	1.213	628.3	689.9	487.8
900	4500	1.352	1.352	1.352	1118.7	1151.9	1229.3
1000	5000	1.512	1.512	1.512	1807.4	1654.3	1610.5

Table 2: Comparison of three variants of IRL_1 methods for least squares loss with $p = 0.1$

Problem		Objective Value			CPU Time		
m	n	IRL_1-1-v	IRL_1-2-v	IRL_1-3-v	IRL_1-1-v	IRL_1-2-v	IRL_1-3-v
100	500	0.174	0.175	0.175	4.5	3.0	0.6
200	1000	0.328	0.337	0.337	12.6	6.9	0.8
300	1500	0.468	0.464	0.464	10.3	29.0	6.2
400	2000	0.630	0.620	0.620	31.3	23.0	5.9
500	2500	0.782	0.791	0.791	47.2	60.0	13.9
600	3000	0.918	0.908	0.910	46.8	154.5	21.3
700	3500	1.048	1.066	1.047	90.5	110.7	32.0
800	4000	1.193	1.215	1.215	88.6	146.5	53.2
900	4500	1.353	1.388	1.388	219.3	199.6	45.5
1000	5000	1.511	1.534	1.513	221.6	242.3	64.8

The samples $\{a^1, \dots, a^m\}$ and the corresponding outcomes b_1, \dots, b_m are generated in the same manner as described in [24]. In detail, for each instance we choose equal number of positive and negative samples, that is, $m_+ = m_- = m/2$, where m_+ (resp., m_-) is the number of samples with outcome $+1$ (resp., -1). The features of positive (resp., negative) samples are independent and identically distributed, drawn from a normal distribution $N(\mu, 1)$, where μ is in turn drawn from a uniform distribution on $[0, 1]$ (resp., $[-1, 0]$). The results of the above methods for the these randomly generated instances with $p = 0.1$ and 0.5 are presented in Tables 5-8, respectively. The CPU time reported here again does not include the time for obtaining initial point x^0 . The similar phenomenon mentioned above can be observed in this experiment.

Table 3: Comparison of three IRL_1 methods for least squares loss with $p = 0.5$

Problem		Objective Value			CPU Time		
m	n	IRL_1 -1	IRL_1 -2	IRL_1 -3	IRL_1 -1	IRL_1 -2	IRL_1 -3
100	500	0.065	0.065	0.065	6.0	5.3	5.3
200	1000	0.106	0.106	0.106	7.8	6.7	6.9
300	1500	0.139	0.139	0.139	29.5	32.2	30.4
400	2000	0.177	0.177	0.177	54.5	60.2	48.5
500	2500	0.217	0.217	0.217	136.9	138.1	118.5
600	3000	0.241	0.241	0.241	219.3	404.5	216.4
700	3500	0.265	0.265	0.265	306.2	465.8	254.3
800	4000	0.299	0.299	0.299	473.8	436.6	557.4
900	4500	0.330	0.330	0.329	612.1	715.7	821.3
1000	5000	0.358	0.358	0.358	974.0	1215.0	877.1

Table 4: Comparison of three variants of IRL_1 methods for least squares loss with $p = 0.5$

Problem		Objective Value			CPU Time		
m	n	IRL_1 -1-v	IRL_1 -2-v	IRL_1 -3-v	IRL_1 -1-v	IRL_1 -2-v	IRL_1 -3-v
100	500	0.065	0.065	0.065	4.4	4.1	4.3
200	1000	0.106	0.106	0.106	5.3	3.9	3.2
300	1500	0.139	0.139	0.139	20.4	13.9	13.8
400	2000	0.176	0.176	0.176	38.2	24.4	27.3
500	2500	0.216	0.217	0.217	74.7	61.0	118.5
600	3000	0.241	0.241	0.241	148.8	94.6	89.8
700	3500	0.265	0.265	0.265	146.2	148.7	144.5
800	4000	0.298	0.300	0.299	409.7	299.4	245.1
900	4500	0.329	0.329	0.329	591.2	423.8	489.6
1000	5000	0.358	0.358	0.358	748.7	273.2	267.1

6 Concluding remarks

In this paper we studied iterative reweighted minimization methods for l_p regularized unconstrained minimization problems (1). In particular, we derived lower bounds for nonzero entries of first- and second-order stationary points, and hence also of local minimizers of (1). We extended some existing IRL_1 and IRL_2 methods to solve (1) and proposed new variants for them. Also, we provided a unified convergence analysis for these methods. In addition, we proposed a novel Lipschitz continuous ϵ -approximation to $\|x\|_p^p$. Using this result, we developed new IRL_1 methods for (1) and showed that any accumulation point of the sequence generated by these methods is a first-order stationary point of problem (1), provided that the approximation parameter ϵ is below a computable threshold value. This is a remarkable result since all existing iterative reweighted minimization methods require that ϵ be dynamically updated and approach to zero. Our computational results demonstrate that the new

Table 5: Comparison of three IRL₁ methods for logistic loss with $p = 0.1$

Problem		Objective Value			CPU Time		
m	n	IRL ₁ -1	IRL ₁ -2	IRL ₁ -3	IRL ₁ -1	IRL ₁ -2	IRL ₁ -3
500	1000	0.082	0.082	0.082	2.8	1.1	0.7
1000	2000	0.102	0.105	0.097	9.3	3.8	7.1
1500	3000	0.152	0.152	0.149	11.5	5.5	6.1
2000	4000	0.166	0.158	0.166	19.8	31.1	13.3
2500	5000	0.204	0.197	0.204	15.4	24.4	10.5
3000	6000	0.234	0.227	0.214	26.6	21.4	58.7
3500	7000	0.246	0.251	0.255	50.1	26.8	36.2
4000	8000	0.262	0.271	0.248	90.2	25.7	36.2
4500	9000	0.265	0.266	0.265	52.5	69.8	40.6
5000	10000	0.278	0.293	0.304	253.3	170.1	121.1

Table 6: Comparison of three variants of IRL₁ methods for logistic loss with $p = 0.1$

Problem		Objective Value			CPU Time		
m	n	IRL ₁ -1-v	IRL ₁ -2-v	IRL ₁ -3-v	IRL ₁ -1-v	IRL ₁ -2-v	IRL ₁ -3-v
500	1000	0.069	0.085	0.090	0.7	0.2	0.1
1000	2000	0.100	0.105	0.097	3.5	1.6	1.3
1500	3000	0.149	0.143	0.146	2.9	2.3	1.4
2000	4000	0.166	0.166	0.166	5.4	7.6	2.6
2500	5000	0.189	0.194	0.184	16.6	11.4	6.0
3000	6000	0.224	0.237	0.231	9.0	20.8	3.5
3500	7000	0.241	0.246	0.245	22.7	16.1	4.4
4000	8000	0.269	0.271	0.266	34.8	24.1	14.7
4500	9000	0.265	0.260	0.238	45.9	37.6	25.9
5000	10000	0.288	0.303	0.293	86.7	23.5	17.3

IRL₁ method and the new variants generally outperform the existing IRL₁ methods [20, 17].

Recently, Zhao and Li [35] proposed an IRL₁ minimization method to identify sparse solutions to undetermined linear systems based on a class of regularizers. When applied to the l_p regularizer, their method becomes one of the first type of IRL₁ methods discussed in Subsection 3.1. Though we only studied the l_p regularized minimization problems, the techniques developed in our paper can be useful for analyzing the iterative reweighted minimization methods for the optimization problems with other regularizers. In addition, most of the results in this paper can be easily generalized to l_p regularized matrix optimization problems.

Table 7: Comparison of three IRL₁ methods for logistic loss with $p = 0.5$

Problem		Objective Value			CPU Time		
m	n	IRL ₁ -1	IRL ₁ -2	IRL ₁ -3	IRL ₁ -1	IRL ₁ -2	IRL ₁ -3
500	1000	0.127	0.127	0.132	2.5	0.7	0.5
1000	2000	0.158	0.158	0.158	5.3	1.9	1.4
1500	3000	0.202	0.203	0.202	8.9	9.2	3.9
2000	4000	0.236	0.230	0.230	15.7	19.3	8.8
2500	5000	0.256	0.256	0.255	12.4	7.8	7.9
3000	6000	0.285	0.280	0.283	24.1	48.1	18.4
3500	7000	0.316	0.316	0.316	29.5	22.9	13.7
4000	8000	0.319	0.319	0.319	50.2	41.4	39.8
4500	9000	0.331	0.333	0.333	69.5	57.5	48.4
5000	10000	0.358	0.358	0.358	94.4	86.2	69.3

Table 8: Comparison of three variants of IRL₁ methods for logistic loss with $p = 0.5$

Problem		Objective Value			CPU Time		
m	n	IRL ₁ -1	IRL ₁ -2	IRL ₁ -3	IRL ₁ -1	IRL ₁ -2	IRL ₁ -3
500	1000	0.127	0.127	0.127	0.2	0.1	0.1
1000	2000	0.157	0.155	0.157	1.2	1.1	0.8
1500	3000	0.206	0.203	0.206	1.8	3.1	1.4
2000	4000	0.230	0.234	0.230	4.0	3.2	4.1
2500	5000	0.254	0.258	0.253	9.1	6.0	5.5
3000	6000	0.282	0.285	0.282	12.5	15.2	9.0
3500	7000	0.312	0.311	0.313	11.5	25.2	12.4
4000	8000	0.326	0.320	0.325	32.2	32.8	12.0
4500	9000	0.333	0.333	0.333	23.7	22.9	20.7
5000	10000	0.359	0.357	0.358	16.9	21.7	15.8

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