SIAM J. OPTIM.
Vol. 15, No. 4, pp. 1147–1154


# A NOTE ON THE LOCAL CONVERGENCE OF A PREDICTOR-CORRECTOR INTERIOR-POINT ALGORITHM FOR THE SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEM BASED ON THE ALIZADEH–HAEBERLY–OVERTON SEARCH DIRECTION*

ZHAOSONG LU† AND RENATO D. C. MONTEIRO†

**Abstract.** This note points out an error in the local quadratic convergence proof of the predictor-corrector interior-point algorithm for solving the semidefinite linear complementarity problem based on the Alizadeh–Haeberly–Overton search direction presented in [M. Kojima, M. Shida, and S. Shindoh, *SIAM J. Optim.*, 9 (1999), pp. 444–465]. Their algorithm is slightly modified and the local quadratic convergence of the resulting method is established.

**Key words.** semidefinite linear complementarity problem, semidefinite programming, interior-point algorithm, predictor-corrector algorithm, local quadratic convergence

**AMS subject classifications.** 90C22, 90C25, 90C30, 65K05

**DOI.** 10.1137/04060531X

## 1. Introduction.

Let $\mathcal{S}$ denote the set of all $n \times n$ symmetric real matrices. Given matrices $X$ and $Y$ in $\Re^{p\times q}$, the standard inner product is defined by $X \bullet Y \equiv \mathrm{tr}(X^T Y)$, where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. The Euclidean norm and its associated operator norm, i.e., the spectral norm, are both denoted by $\|\cdot\|$. The Frobenius norm of a $p \times q$-matrix $X$ is defined as $\|X\|_F \equiv (X \bullet X)^{1/2}$. If $X \in \mathcal{S}$ is positive semidefinite (resp., definite), we write $X \succeq 0$ (resp., $X \succ 0$). The cone of positive semidefinite (resp., definite) matrices is denoted by $\mathcal{S}_+$ (resp., $\mathcal{S}_{++}$). The identity matrix will be denoted by $I$.

Let $\mathcal{F}$ be a $n(n+1)/2$-dimensional affine subspace of $\mathcal{S} \times \mathcal{S}$, and

$$\mathcal{F}_+ = \{(X,Y) \in \mathcal{F} : X \succeq 0, Y \succeq 0\}.$$

We are concerned with the semidefinite linear complementarity problem (SDLCP):

$$\text{find a } (X,Y) \in \mathcal{F}_+ \text{ such that } X \bullet Y = 0. \tag{1.1}$$

We call a $(X,Y) \in \mathcal{F}_+$ a feasible solution of the SDLCP (1.1). Throughout this note we assume the monotonicity of the affine subspace $\mathcal{F}$:

$$(U' - U) \bullet (V' - V) \geq 0 \text{ for every } (U',V'), (U,V) \in \mathcal{F}.$$

Kojima, Shida, and Shindoh [3] have proposed a globally convergent Mizuno–Todd–Ye-type predictor-corrector infeasible-interior-point algorithm (Algorithm 2.1 of [3]), with the use of the Alizadeh–Haeberly–Overton (AHO) search direction, for the monotone SDLCP (1.1), and demonstrated its local quadratic convergence under the strict complementarity condition.

*Received by the editors March 16, 2004; accepted for publication (in revised form) November 24, 2004; published electronically August 3, 2005.
http://www.siam.org/journals/siopt/15-4/60531.html

†School of ISyE, Georgia Tech, Atlanta, GA 30332 (zhaosong@isye.gatech.edu, monteiro@isye.gatech.edu). The second author was supported in part by NSF grants CCR-0203113 and INT-9910084 and ONR grant N00014-03-1-0401.

This note has two purposes. One is to point out an error in the proof of the local quadratic convergence of the algorithm presented in [3]. The other is to describe a modified variant of this method and establish its local quadratic convergence.

This note is organized as follows. In section 2, we describe the algorithm presented in [3] and point out an error made in [3] on the proof of its local quadratic convergence. In section 3, we describe a slight modification of this algorithm and establish the local quadratic convergence of the resulting method.

**1.1. Notation.** Given functions $f : \Omega \to E$ and $g : \Omega \to \Re_{++}$, where $\Omega$ is an arbitrary set and $E$ is a normed vector space, and a subset $\tilde{\Omega} \subset \Omega$, we write $f(w) = \mathcal{O}(g(w))$ for all $w \in \tilde{\Omega}$ to mean that there exists a constant $M > 0$ such that $\|f(w)\| \le Mg(w)$ for all $w \in \tilde{\Omega}$; moreover, for a function $U : \Omega \to \mathcal{S}_{++}$, we write $U(w) = \Theta(g(w))$ for all $w \in \tilde{\Omega}$ if $U(w) = \mathcal{O}(g(w))$ and $U(w)^{-1} = \mathcal{O}(1/g(w))$ for all $w \in \tilde{\Omega}$. The latter condition is equivalent to the existence of a constant $M > 0$ such that

$$\frac{1}{M} I \preceq \frac{1}{g(w)} U(w) \preceq MI \quad \forall w \in \Omega.$$

**2. A predictor-corrector interior-point algorithm.** In this section, we describe the predictor-corrector infeasible-interior-point algorithm using AHO search direction (Algorithm 2.1 in [3]) for monotone SDLCP (1.1), and point out an error in the proof of its local quadratic convergence in Theorem 5.1 of [3].

Throughout this note we use the same notation as in [3],

$$\begin{aligned} \zeta &: \ \text{a constant not less than } 1/\sqrt{n}, \\ \mathcal{F}_0 &= \{(U', V') - (U, V) : (U', V'), (U, V) \in \mathcal{F}\}, \\ \tilde{\mathcal{N}}(\gamma, \tau) &= \left\{ (X, Y) \in \mathcal{S}_+ \times \mathcal{S}_+ : \begin{array}{l} (XY + YX)/2 \succeq (1-\gamma)\tau I, \\ X \bullet Y/n \le (1 + \zeta\gamma)\tau \end{array} \right\}, \end{aligned}$$

for each $\gamma \in [0, 1]$ and each $\tau \ge 0$.

Before describing Algorithm 2.1 of [3], we recall Hypothesis 2.1 of [3].

*Hypothesis* 2.1 *of* [3]. Let $\omega^* \ge 1$. There exists a solution $(X^*, Y^*)$ of SDLCP (1.1) such that

$$\omega^* X^0 \succeq X^* \text{ and } \omega^* Y^0 \succeq Y^*.$$

For notational convenience, we introduce one operator as follows:

$$H_I(M) = \frac{M + M^T}{2} \quad \forall M \in \Re^{n \times n}.$$

We are ready to describe Algorithm 2.1 of [3] as follows.

ALGORITHM 2.1 OF [3].

*Step* 0. Choose an accuracy parameter $\epsilon \ge 0$, a neighborhood parameter $\gamma \in (0, 1)$, and an initial point $(X^0, Y^0) = (\sqrt{\mu^0} I, \sqrt{\mu^0} I)$ with some $\mu^0 > 0$. Let $\theta^0 = 1$, $\sigma = 2\omega^*/(1-\gamma) + 1$, $\gamma^0 = 0$, and $k = 0$.

*Step* 1. If the inequality

$$\theta^k (X^0 \bullet Y^k + X^k \bullet Y^0) \le \sigma X^k \bullet Y^k$$

does not hold, then stop.

*Step* 2 (predictor step). Compute a solution $(dX_p^k, dY_p^k)$ of the system of equations

$$\begin{aligned} &H_I(X^k dY_p^k + dX_p^k Y^k) = -H_I(X^k Y^k), \\ &(X^k + dX_p^k, Y^k + dY_p^k) \in \mathcal{F}. \end{aligned}$$

Let

$$\begin{aligned} \delta_p^k &= \frac{\|dX_p^k\|_F \|dY_p^k\|_F}{\theta^k \mu^0}, \\ \hat{\alpha}_p^k &= \frac{2}{\sqrt{1 + 4\delta_p^k/(\gamma - \gamma^k)} + 1}, \\ \check{\alpha}_p^k &= \max \left\{ \alpha' \in [0,1] : \begin{array}{l} (X^k + \alpha dX_p^k, Y^k + \alpha dY_p^k) \\ \in \tilde{\mathcal{N}}(\gamma, (1-\alpha)\theta^k \mu^0) \\ \text{for every } \alpha \in [0, \alpha'] \end{array} \right\}. \end{aligned}$$

Choose a step length $\alpha_p^k \in [\hat{\alpha}_p^k, \check{\alpha}_p^k]$. Let

$$(X_c^k, Y_c^k) = (X^k, Y^k) + \alpha_p^k (dX_p^k, dY_p^k) \text{ and } \theta^{k+1} = (1 - \alpha_p^k)\theta^k.$$

*Step* 3. If $\theta^{k+1} \le \epsilon$, then stop. If the inequality

$$\theta^{k+1}(X^0 \bullet Y_c^k + X_c^k \bullet Y^0) \le \sigma X_c^k \bullet Y_c^k \tag{2.1}$$

does not hold, then stop.

*Step* 4 (corrector step). Compute a solution $(dX_c^k, dY_c^k)$ of the system of equations

$$\left\{ \begin{array}{l} H_I(X_c^k dY_c^k + dX_c^k Y_c^k) = \theta^{k+1}\mu^0 I - H_I(X_c^k Y_c^k), \\ (dX_c^k, dY_c^k) \in \mathcal{F}_0. \end{array} \right. \tag{2.2}$$

Let

$$\begin{aligned} \delta_c^k &= \frac{\|dX_c^k\|_F \|dY_c^k\|_F}{\theta^{k+1}\mu^0}, \\ \hat{\alpha}_c^k &= \left\{ \begin{array}{ll} \gamma/(2\delta_c^k) & \text{if } \gamma \le 2\delta_c^k, \\ 1 & \text{if } \gamma > 2\delta_c^k, \end{array} \right. \\ \check{\gamma}^{k+1} &= \left\{ \begin{array}{ll} \gamma(1 - \gamma/(4\delta_c^k)) & \text{if } \gamma \le 2\delta_c^k, \\ \delta_c^k & \text{if } \gamma > 2\delta_c^k, \end{array} \right. \\ \hat{\gamma}^{k+1} &= \min \left\{ \gamma' \in [0,1] : \begin{array}{l} (X_c^k + \alpha dX_c^k, Y_c^k + \alpha dY_c^k) \\ \in \tilde{\mathcal{N}}(\gamma', \theta^{k+1}\mu^0) \\ \text{for some } \alpha \in [0,1] \end{array} \right\}. \end{aligned} \tag{2.3}$$

Choose a step length $\alpha_c^k \in [0,1]$ and $\gamma^{k+1}$ such that

$$\begin{aligned} &\hat{\gamma}^{k+1} \le \gamma^{k+1} \le \check{\gamma}^{k+1}, \\ &(X_c^k + \alpha_c^k dX_c^k, Y_c^k + \alpha_c^k dY_c^k) \in \tilde{\mathcal{N}}(\gamma^{k+1}, \theta^{k+1}\mu^0). \end{aligned}$$

(It has been shown in Lemma 3.8 of [3] that the pair of $\alpha_c^k = \hat{\alpha}_c^k$ and $\gamma^{k+1} = \check{\gamma}^{k+1}$ satisfies the relation above.) Let $(X^{k+1}, Y^{k+1}) = (X_c^k, Y_c^k) + \alpha_c^k (dX_c^k, dY_c^k)$.

*Step* 5. Replace $k$ by $k+1$. Go to Step 1.

Before ending this section, we remark that the proof of Theorem 5.1 (local convergence theorem) of [3] is not correct since it is based on the claim that $\delta_c^k = \mathcal{O}(1)$, which in turn was incorrectly established in the proof of this result. Indeed, in the first two lines of the proof of Theorem 5.1 of [3], the authors claimed that $\delta_c^k = \mathcal{O}(1)$ holds by (iii) of Lemma 3.1, the definition of $\delta_c^k$, and the fact that $2\theta^k I - (X^k Y^k + Y^k X^k) = \mathcal{O}(\theta^k)$. However, from those arguments we can only conclude $\delta_c^k = \mathcal{O}(\frac{1}{\theta^{k+1}})$. Let us investigate this proof in more detail. From Step 2 of Algorithm 2.1 of [3], we see that

$$(X_c^k, Y_c^k) \in \tilde{\mathcal{N}}(\gamma, \theta^{k+1}\mu^0), \tag{2.4}$$

which, together with (2.1) and Lemma 3.4 of [3], implies that $(X_c^k, Y_c^k) = \mathcal{O}(1)$. Also, by (2.4) and Lemma 3.1 (i) of [3], we have $H_I(X_c^k Y_c^k) = \mathcal{O}(\theta^{k+1})$, which implies that

$$\theta^{k+1}\mu^0 I - H_I(X_c^k Y_c^k) = \mathcal{O}(\theta^{k+1}). \tag{2.5}$$

Now, using (2.4), (2.2), (2.5); Lemma 3.1 (iii) of [3]; and the fact $(X_c^k, Y_c^k) = \mathcal{O}(1)$, we have

$$\|dX_c^k\|_F \le \frac{2\|X_c^k\|_F \; \|\theta^{k+1}\mu^0 I - H_I(X_c^k Y_c^k)\|_F}{(1-\gamma)\theta^{k+1}\mu^0} = \mathcal{O}(1). \tag{2.6}$$

Similarly, we have $\|dY_c^k\|_F = \mathcal{O}(1)$, which together with (2.6) and (2.3) implies that $\delta_c^k = \mathcal{O}(\frac{1}{\theta^{k+1}})$. Due to this and [2], we believe that the claim $\delta_c^k = \mathcal{O}(1)$ does not hold for general SDLCPs, even though it holds under a suitable nondegeneracy assumption on the SDLCP, namely Condition 6.1 of [3] (see the proof in section 6 of [3]). Hence, Algorithm 2.1 of [3] can only be claimed to be locally quadratically convergent for nondegenerate SDLCPs. In the next section, we will describe a slight modification of Algorithm 2.1 of [3] which is locally quadratically convergent.

**3. Slightly modified algorithm.** In this section, we describe a slight modification of Algorithm 2.1 of [3] and establish its local quadratic convergence.

The modified algorithm is the same as before except that the definition of $\delta_c^k$ in (2.3) is replaced by

$$\delta_c^k = \frac{\|dX_c^k dY_c^k\|_F}{\theta^{k+1}\mu^0}. \tag{3.1}$$

Accordingly, we refer to the modified algorithm as Algorithm 2.1′. Our main effort from now on will be to establish that the quantity $\delta_c^k$, as defined in (3.1), has the property that $\delta_c^k = \mathcal{O}(1)$.

First, we will argue that Algorithm 2.1′ is globally convergent. It can be shown that Lemmas 3.1–3.7 of [3] also hold for Algorithm 2.1′. The next result shows that Lemma 3.8 also holds for Algorithm 2.1′ if $\zeta \ge 1/\sqrt{n}$.

LEMMA 3.1. *For Algorithm* 2.1′, *if* $\zeta \ge 1/\sqrt{n}$, *Lemma* 3.8 *in* [3] *holds, where* $\zeta$ *is a constant defined at the beginning of section* 2 *of* [3].

*Proof.* Using the fact that $H_I(dX_c^k dY_c^k) \ge -\|dX_c^k dY_c^k\|_F I$ and $dX_c^k \bullet dY_c^k \le \sqrt{n}\|dX_c^k dY_c^k\|_F$, and the condition $\zeta \ge 1/\sqrt{n}$, we can show that the conclusion holds in a similar way as the proof given in Lemma 3.8 of [3]. ∎

Using Lemmas 3.1–3.7 of [3] and Lemma 3.1 and following the same proof as the one given in Theorem 2.1 of [3], we see that Theorem 2.1 (global convergence theorem) in [3] also holds for Algorithm 2.1′; namely, Algorithm 2.1′ is globally convergent.

We will now show that Algorithm 2.1′ is locally quadratically convergent under the following standard condition commonly used in the local convergence analysis of interior-point algorithms for SDLCP.

*Condition* 5.1 of [3] (strict complementarity). There is a solution $(X^*, Y^*)$ of SDLCP (1.1) such that $X^* + Y^* \succ 0$.

We next state and prove some technical results. The first one is due to Monteiro and Tsuchiya [4].

LEMMA 3.2 (Lemma 2.1 of [4]). *For every $A \in \mathcal{S}_{++}$ and $H \in \mathcal{S}$, the equation $AU + UA = H$ has a unique solution $U \in \mathcal{S}$. Moreover, this solution satisfies $\|AU\|_F \le \|H\|_F/\sqrt{2}$.*

Under Condition 5.1 of [3], we have a solution $(X^*, Y^*)$ of the SDLCP (1.1) satisfying $X^* + Y^* \succ 0$. Since $X^*$ and $Y^*$ commute, there exists an orthogonal matrix $Q$ such that

$$Q^T X^* Q = \begin{pmatrix} \Lambda_B & 0 \\ 0 & 0 \end{pmatrix}, \quad Q^T Y^* Q = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_N \end{pmatrix},$$

where $\Lambda_B$ and $\Lambda_N$ are positive diagonal matrices with dimension $m$ and $n-m$ for some $m \in \{0, 1, 2, \ldots, n\}$, respectively. For each $(X, Y) \in \mathcal{S} \times \mathcal{S}$, define the following optimal partition:

$$Q^T X Q \equiv \hat{X} = \begin{pmatrix} \hat{X}_B & \hat{X}_J \\ \hat{X}_J^T & \hat{X}_N \end{pmatrix}, \quad Q^T Y Q \equiv \hat{Y} = \begin{pmatrix} \hat{Y}_B & \hat{Y}_J \\ \hat{Y}_J^T & \hat{Y}_N \end{pmatrix}.$$

LEMMA 3.3. *Assume that $(X, Y) \in \tilde{\mathcal{N}}(\gamma, \tau)$. Let $(dX(\tau), dY(\tau))$ be a solution of the system of equations*

$$H_I(dX\ Y + X\ dY) = \tau I - H_I(XY), \quad (dX, dY) \in \mathcal{F}_0, \tag{3.2}$$

*$\widehat{dX} \equiv Q^T dX Q$ and $\widehat{dY} \equiv Q^T dY Q$. Under Condition* 5.1 *of* [3] *and $\zeta \ge 1/\sqrt{n}$, there then holds*

$$\widehat{dX}_B(\tau) = \mathcal{O}(1), \quad \widehat{dX}_N(\tau) = \mathcal{O}(\tau), \tag{3.3}$$

$$\widehat{dY}_B(\tau) = \mathcal{O}(\tau), \quad \widehat{dY}_N(\tau) = \mathcal{O}(1). \tag{3.4}$$

*Proof.* For notational convenience, we will use $\widehat{dX}$ and $\widehat{dY}$ to denote $\widehat{dX}(\tau)$ and $\widehat{dY}(\tau)$, respectively. Using Lemmas 5.3 and 5.5 of [3], we have

$$\hat{X} = Q^T X Q = \begin{pmatrix} \Theta(1) & \mathcal{O}(\tau) \\ \mathcal{O}(\tau) & \mathcal{O}(\tau) \end{pmatrix}, \tag{3.5}$$

$$\hat{Y} = Q^T Y Q = \begin{pmatrix} \mathcal{O}(\tau) & \mathcal{O}(\tau) \\ \mathcal{O}(\tau) & \Theta(1) \end{pmatrix}. \tag{3.6}$$

This immediately implies that $X = \mathcal{O}(1)$ and $Y = \mathcal{O}(1)$. In view of Lemma 3.1 (i) of [3] and the definition of $H_I(\cdot)$, we immediately see that

$$\tau I - H_I(XY) = \mathcal{O}(\tau). \tag{3.7}$$

Letting $C = 2(\tau I - H_I(XY))$ and using Lemma 3.1 (iii) of [3], we obtain that

$$\|\widehat{dX}\|_F = \|dX\|_F \le \frac{\|X\|_F \|C\|_F}{(1-\gamma)\tau} = \mathcal{O}(1).$$

Hence, $\widehat{dX} = \mathcal{O}(1)$. Similarly, we can show that $\widehat{dY} = \mathcal{O}(1)$. Note that the system (3.2) can be written as

$$H_I(\widehat{dX}\ \hat{Y} + \hat{X}\ \widehat{dY}) = \tau I - H_I(\hat{X}\hat{Y}), \quad (\widehat{dX}, \widehat{dY}) \in \hat{\mathcal{F}}_0, \tag{3.8}$$

where $\hat{\mathcal{F}}_0 \equiv \{M = Q^T P Q : P \in \mathcal{F}_0\}$. From (3.7), we easily see that

$$\tau I - H_I(\hat{X}\hat{Y}) = \mathcal{O}(\tau). \tag{3.9}$$

Using this fact and (3.8), we obtain that

$$H_I(\widehat{dX}_B \hat{Y}_B + \widehat{dX}_J \hat{Y}_J^T + \hat{X}_B \widehat{dY}_B + \hat{X}_J \widehat{dY}_J^T) = \mathcal{O}(\tau). \tag{3.10}$$

Using (3.5), (3.6), (3.10), and the fact that $\widehat{dX} = \mathcal{O}(1)$ and $\widehat{dY} = \mathcal{O}(1)$, we have

$$H_I(\hat{X}_B \widehat{dY}_B) = \mathcal{O}(\tau),$$

which together with (3.5) and Lemma 3.2 implies $\widehat{dY}_B = \mathcal{O}(\tau)$. We can show that $\widehat{dX}_N = \mathcal{O}(\tau)$ in a similar way. □

LEMMA 3.4. *Assume that* $(X, Y) \in \tilde{\mathcal{N}}(\gamma, \tau)$. *Let* $(\widehat{dX}(\tau), \widehat{dY}(\tau))$ *be defined in Lemma* 3.3. *Under Condition* 5.1 *of* [3] *and* $\zeta \geq 1/\sqrt{n}$, *there then holds*

$$\|\widehat{dY}_J(\tau)\| = \Theta(\|\widehat{dX}_J(\tau)\|) + \mathcal{O}(\tau),$$
$$-\widehat{dX}_J(\tau) \bullet \widehat{dY}_J(\tau) = \Theta(\|\widehat{dX}_J(\tau)\|^2) + \mathcal{O}(\tau\|\widehat{dX}_J(\tau)\|).$$

*Proof*. For notational convenience, we will use $\widehat{dX}$ and $\widehat{dY}$ to denote $\widehat{dX}(\tau)$ and $\widehat{dY}(\tau)$, respectively. Using (3.9) and (3.8), we obtain that

$$\begin{aligned}&\widehat{dX}_B \hat{Y}_J + \widehat{dX}_J \hat{Y}_N + \hat{X}_B \widehat{dY}_J + \hat{X}_J \widehat{dY}_N \\ &+ \widehat{dY}_B \hat{X}_J + \widehat{dY}_J \hat{X}_N + \hat{Y}_B \widehat{dX}_J + \hat{Y}_J \widehat{dX}_N = \mathcal{O}(\tau).\end{aligned}$$

This identity together with (3.5), (3.6), (3.3), and (3.4) implies that

$$\widehat{dX}_J \hat{Y}_N + \hat{X}_B \widehat{dY}_J = \mathcal{O}(\tau). \tag{3.11}$$

Using this identity, we obtain that

$$\widehat{dY}_J = -\hat{X}_B^{-1}(\widehat{dX}_J \hat{Y}_N - \mathcal{O}(\tau)). \tag{3.12}$$

Using this identity ((3.5) and (3.6)), we see that the first conclusion follows. Using (3.12), (3.5), and (3.6), we obtain that

$$\begin{aligned}\widehat{dX}_J \bullet \widehat{dY}_J &= -\mathrm{tr}(\widehat{dX}_J^T \hat{X}_B^{-1} \widehat{dX}_J \hat{Y}_N) + \mathcal{O}(\tau\|\widehat{dX}_J\|) \\ &= -\|(\hat{X}_B)^{-1/2} \widehat{dX}_J (\hat{Y}_N)^{1/2}\|_F^2 + \mathcal{O}(\tau\|\widehat{dX}_J\|),\end{aligned}$$

which together with (3.5) and (3.6) implies the second conclusion. □

LEMMA 3.5. *Assume that* $(X, Y) \in \tilde{\mathcal{N}}(\gamma, \tau)$. *Let* $(\widehat{dX}(\tau), \widehat{dY}(\tau))$ *be defined in Lemma* 3.3. *Under Condition* 5.1 *of* [3] *and* $\zeta \geq 1/\sqrt{n}$, *there then holds*

$$\widehat{dX}_J(\tau) = \mathcal{O}(\tau), \quad \widehat{dY}_J(\tau) = \mathcal{O}(\tau).$$

*Proof.* Suppose that $\|\widehat{dX}_J(\tau)\| = \mathcal{O}(\tau)$ does not hold. Then there exists a sequence $\tau_k \downarrow 0$ as $k \to \infty$ such that $\tau_k = o(\|\widehat{dX}_J(\tau_k)\|)$. For convenience, we omit the index $k$ from $\tau_k$ throughout the remaining proof. Then the above identity can be written as $\tau = o(\|\widehat{dX}_J(\tau)\|)$, which together with Lemma 3.4 implies that

$$\|\widehat{dY}_J(\tau)\| = \Theta(\|\widehat{dX}_J(\tau)\|), \tag{3.13}$$

$$-\widehat{dX}_J(\tau) \bullet \widehat{dY}_J(\tau) = \Theta(\|\widehat{dX}_J(\tau)\|^2). \tag{3.14}$$

For any $\tau > 0$, consider the linear system

$$(\widehat{dX}, \widehat{dY}) - (\widehat{dX}(\tau), \widehat{dY}(\tau)) \in \hat{\mathcal{F}}_0,$$

$$\widehat{dX}_J - \widehat{dX}_J(\tau) = -\widehat{dX}_J(\tau), \tag{3.15}$$

$$\widehat{dX}_N - \widehat{dX}_N(\tau) = -\widehat{dX}_N(\tau), \tag{3.16}$$

$$\widehat{dY}_J - \widehat{dY}_J(\tau) = -\widehat{dY}_J(\tau), \tag{3.17}$$

$$\widehat{dY}_B - \widehat{dY}_B(\tau) = -\widehat{dY}_B(\tau). \tag{3.18}$$

We see that any $(\widehat{dX}, \widehat{dY}) = (0, 0)$ is a feasible solution to this system. Hence, by Hoffman lemma [1] (see also Lemma A.3, p. 248 of [5]), there exists a sufficiently large constant $\hat{C}$ (independent on $\tau$) such that for any $\tau > 0$, this system has a solution $(\overline{dX}, \overline{dY}) \in \mathcal{S} \times \mathcal{S}$ (dependent on $\tau$) such that

$$\|(\overline{dX}, \overline{dY}) - (\widehat{dX}(\tau), \widehat{dY}(\tau))\| \le \hat{C}(\|\widehat{dX}_N(\tau)\| + \|\widehat{dY}_B(\tau)\| + \|\widehat{dX}_J(\tau)\| + \|\widehat{dY}_J(\tau)\|). \tag{3.19}$$

Obviously, the monotonicity holds for $\hat{\mathcal{F}}_0$ due to the monotonicity of $\mathcal{F}_0$. Hence, we have

$$(\overline{dX} - \widehat{dX}(\tau)) \bullet (\overline{dY} - \widehat{dY}(\tau)) \ge 0.$$

Hence, it follows that

$$-(\overline{dX}_B - \widehat{dX}_B(\tau)) \bullet \widehat{dY}_B(\tau) + 2\widehat{dX}_J(\tau) \bullet \widehat{dY}_J(\tau) - \widehat{dX}_N(\tau) \bullet (\overline{dY}_N - \widehat{dY}_N(\tau)) \ge 0. \tag{3.20}$$

Note that $\|\overline{dX}_B - \widehat{dX}_B(\tau)\| \le \|\overline{dX} - \widehat{dX}(\tau)\|$ and $\|\overline{dY}_N - \widehat{dY}_N(\tau)\| \le \|\overline{dY} - \widehat{dY}(\tau)\|$. Using this fact, (3.20), (3.15)–(3.18), (3.19), (3.13), (3.14), (3.3), and (3.4), we obtain that, for all $\tau > 0$ sufficiently small,

$$\begin{aligned}
|\widehat{dX}_J(\tau) \bullet \widehat{dY}_J(\tau)| &\le \frac{1}{2}\left|(\overline{dX}_B - \widehat{dX}_B(\tau)) \bullet \widehat{dY}_B(\tau) + \widehat{dX}_N(\tau) \bullet (\overline{dY}_N - \widehat{dY}_N(\tau))\right| \\
&\le \check{C}\tau(\|\overline{dX} - \widehat{dX}(\tau)\| + \|\overline{dY} - \widehat{dY}(\tau)\|) \\
&\le 2\check{C}\hat{C}\tau(\|\widehat{dX}_N(\tau)\| + \|\widehat{dY}_B(\tau)\| + \|\widehat{dX}_J(\tau)\| + \|\widehat{dY}_J(\tau)\|) \\
&\le \tilde{C}\tau\left(\tau + \sqrt{|\widehat{dX}_J(\tau) \bullet \widehat{dY}_J(\tau)|}\right),
\end{aligned}$$

where $\check{C}$ and $\tilde{C}$ are some constants and the last inequality follows from (3.13) and (3.14). Let $\xi = \sqrt{|\widehat{dX}_J(\tau) \bullet \widehat{dY}_J(\tau)|}$. From the last inequality above, we have

$\xi^2 \leq \tilde{C}\tau(\tau+\xi)$, which together with the fact $\xi > 0$ implies $\xi \leq (\tilde{C} + \sqrt{5\tilde{C}})\tau/2$. Hence, $\xi = \mathcal{O}(\tau)$. Using this result and (3.14), we obtain $\|\widehat{dX}_J(\tau)\| = \mathcal{O}(\tau)$, which contradicts with the assumption $\tau = o(\|\widehat{dX}_J(\tau)\|)$. Therefore, $\|\widehat{dX}_J(\tau)\| = \mathcal{O}(\tau)$ holds. The proof of $\|\widehat{dY}_J(\tau)\| = \mathcal{O}(\tau)$ immediately follows from Lemma 3.4. □

We are now in a position to state the main result of this section, which establishes the local quadratic convergence of Algorithm 2.1′.

THEOREM 3.6. *Assume that Hypothesis* 2.1 *and Condition* 5.1 *of* [3] *hold. If* $\zeta \geq 1/\sqrt{n}$, *Theorem* 5.1 *(local convergence theorem) of* [3] *holds for Algorithm* 2.1′.

*Proof.* Since $(dX_c^k, dY_c^k)$ satisfies (2.2), it implies that $(dX_c^k, dY_c^k)$ also satisfies the system (3.2) with $\tau = \theta^{k+1}\mu^0$. We also know that $(X_c^k, Y_c^k) \in \tilde{\mathcal{N}}(\gamma, \tau)$. Hence, in view of Lemmas 3.3 and 3.5, we have

$$\widehat{dX}_c^k = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\theta^{k+1}) \\ \mathcal{O}(\theta^{k+1}) & \mathcal{O}(\theta^{k+1}) \end{pmatrix},$$
$$\widehat{dY}_c^k = \begin{pmatrix} \mathcal{O}(\theta^{k+1}) & \mathcal{O}(\theta^{k+1}) \\ \mathcal{O}(\theta^{k+1}) & \mathcal{O}(1) \end{pmatrix}.$$

It implies that

$$\delta_c^k = \frac{\|dX_c^k dY_c^k\|_F}{\theta^{k+1}\mu^0} = \frac{\|\widehat{dX}_c^k \widehat{dY}_c^k\|_F}{\theta^{k+1}\mu^0} = \mathcal{O}(1).$$

The remaining part of the proof is based on similar arguments as the ones used in the proof of Theorem 5.1 of [3]. □

## REFERENCES

[1] A. J. HOFFMAN, *On approximate solutions of systems of linear inequalities*, J. Research Nat. Bur. Standards, 49 (1952), pp. 263–265.
[2] M. KOJIMA, *private communication*, 2003.
[3] M. KOJIMA, M. SHIDA, AND S. SHINDOH, *A predictor-corrector interior-point algorithm for the semidefinite linear complementarity problem using the Alizadeh–Haeberly–Overton search direction*, SIAM J. Optim., 9 (1999), pp. 444–465.
[4] R. D. C. MONTEIRO AND T. TSUCHIYA, *Polynomial convergence of a new family of primal-dual algorithms for semidefinite programming*, SIAM J. Optim., 9 (1999), pp. 551–577.
[5] S. WRIGHT, *Primal-Dual Interior-Point Methods*, SIAM, Philadelphia, 1997.