Error Bounds for First- and Second-Order Approximations of Eigenvalues and Singular Values

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Abstract

We derive explicit error bounds for first- and second-order approximations for all eigenvalues of a real symmetric matrix by use of techniques from functional calculus of linear operators. We also apply these results to obtain error bounds for first- and second-order approximations for singular values of a real matrix. These error bounds are potentially useful for designing algorithms for solving eigenvalue and singular value optimization problems.

Key words. Error bound, directional derivative, first-order approximation, second-order approximation

1 Introduction

Optimization of eigenvalues of real symmetric matrices arises in many applications such as structural optimization problems in mechanics (see, for example, [4]) and graph-partitioning problems (see, for example, [2]). We refer interested readers to [12] for discussion of more applications. As mentioned in [9], sensitivity analysis of eigenvalues plays essential role in developing efficient algorithms for eigenvalue optimization. This topic has attracted considerable research interest (see, for example, [3–5, 7–11, 13–15, 17]). Below we only briefly mention several results most relevant to our work in this paper.

First, it is well-known that when the mth largest eigenvalue $\lambda_m(X_0)$ has multiplicity one, $\lambda_m(X)$ is an analytic function of X at X_0 (see, for example, [7]). On the other hand, when the multiplicity is not one, $\lambda_m(X)$ is not differentiable at X_0. Nevertheless, Hiriart-Urruty and Ye [5] showed that the first-order directional derivative always exists for all eigenvalues of symmetric matrices, regardless of multiplicity, i.e., the limit

$$\lim_{t\to 0^+} \frac{\lambda_m(X+t\Delta)-\lambda_m(X)}{t} =: \lambda_m'(X;\Delta) \tag{1}$$

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exists for all $1 \le m \le n$ and $X,\ \Delta \in \mathcal{S}^n$, where $\mathcal{S}^n$ denotes the set of all $n \times n$ real symmetric matrices. They also provided an explicit expression for $\lambda'_m(X;\Delta)$. Later, Torki [17] made use of a perturbation result about invariant subspaces from [16] and showed that the second-order directional derivative also exists, i.e., the limit

$$\lim_{t\to 0^+} \frac{\lambda_m(X+t\Delta)-\lambda_m(X)-t\lambda'_m(X;\Delta)}{\frac{1}{2}t^2} =: \lambda''_m(X;\Delta) \tag{2}$$

exists all $1 \le m \le n$ and $X,\ \Delta \in \mathcal{S}^n$. And an explicit formula for $\lambda''_m(X;\Delta)$ was also given.

In this paper, we establish error bounds for first- and second-order approximations of eigenvalues of $X \in \mathcal{S}^n$ by explicitly choosing constants δ, C_1, C_2 and a matrix Q in terms of X and m such that

$$|\lambda_m(X+\Delta)-\lambda_m(X)-\lambda'_m(X;\Delta)| \le C_1\|\Delta\|^2,$$

$$|\lambda_m(X+\Delta)-\lambda_m(X)-\lambda'_m(X;\Delta+\Delta Q\Delta)| \le C_2\|\Delta\|^3$$

whenever $\Delta \in \mathcal{S}^n$ and $\|\Delta\| < \delta$. The results (1) and (2) can thus be obtained as byproducts. We also apply these results to obtain error bounds for first- and second-order approximations of singular values. Those bounds are potentially useful for designing algorithms for solving eigenvalue and singular value optimization problems. Our proof techniques differ much from those in previous works in that we make extensive use of the integral representation of a linear operator involving the resolvent as in [1].

The rest of this paper is organized as follows. In Section 2, we introduce some notations and establish some technical preliminaries. In Section 3, we derive error bounds for first- and second-order approximations of eigenvalues, respectively. Finally, we apply these results to obtain error bounds for first- and second-order approximations of singular values in Section 4.

2 Notations and technical preliminaries

All spaces of this paper are for real vectors or matrices unless explicitly stated otherwise. Let $\Re^n$ denote the n-dimensional Euclidean space. For a vector $v \in \Re^n$, the Euclidean norm of v is denoted by $\|v\|$, and $\mathrm{Diag}(v)$ denotes a diagonal matrix with v along its diagonal. Let $\mathcal{S}^n$ denote the space of all $n \times n$ symmetric matrices. For any $X \in \mathcal{S}^n$, all n eigenvalues of X are denoted by $\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_n(X)$. Let $\Re^{p\times q}$ denote the space of all $p \times q$ matrices. For a $Z \in \Re^{p\times q}$, the spectral norm of Z is denoted by $\|Z\|$. The identity matrix is denoted by I, whose dimension should be clear from the context.

For an $X \in \mathcal{S}^n$ and each $1 \le m \le n$, we define the integers i_m and j_m as the number of eigenvalues ranking before m that equal $\lambda_m(X)$ and the number of eigenvalues ranking strictly after m that equal $\lambda_m(X)$, respectively. Hence,

$$\begin{aligned}\lambda_1(X) &\ge \cdots \ge \lambda_{m-i_m}(X) > \lambda_{m-i_m+1}(X) = \cdots = \lambda_m(X)\\ &= \lambda_{m+1}(X) = \cdots = \lambda_{m+j_m}(X) > \lambda_{m+j_m+1}(X) \ge \cdots \ge \lambda_n(X).\end{aligned}$$

In addition, we define $\lambda_0(X) = \infty$, $\lambda_{n+1}(X) = -\infty$, and

$$r_m := \frac{1}{2}\min\{\lambda_{m-i_m}(X)-\lambda_m(X), \lambda_m(X)-\lambda_{m+j_m+1}(X)\}. \tag{3}$$

We can immediately observe that $r_m > 0$.

The following lemmas will be used subsequently. The first lemma can be found in [6, Page 198].

Lemma 1. *If $X, \Delta \in \mathcal{S}^n$, then*

$$|\lambda_i(X+\Delta) - \lambda_i(X)| \leq \|\Delta\| \quad \forall i = 1, ..., n.$$

Lemma 2. *If $\Delta \in \mathcal{S}^n$ and $\|\Delta\| < r_m/2$, then*

$$|\lambda_i(X+\Delta) - \lambda_m(X)| > \frac{3r_m}{2} \quad \forall i \notin \{m - i_m + 1, ..., m, ..., m + j_m\}, \tag{4}$$

$$|\lambda_i(X+\Delta) - \lambda_m(X)| < \frac{r_m}{2} \quad \forall i \in \{m - i_m + 1, ..., m, ..., m + j_m\}. \tag{5}$$

Proof. First, from Lemma 1 and the assumption on Δ, we see that

$$\lambda_{m-i_m}(X+\Delta) \;\geq\; \lambda_{m-i_m}(X) - \|\Delta\| \;>\; \lambda_{m-i_m}(X) - \frac{r_m}{2}, \tag{6}$$

$$\lambda_{m+j_m+1}(X+\Delta) \;\leq\; \lambda_{m+j_m+1}(X) + \|\Delta\| \;<\; \lambda_{m+j_m+1}(X) + \frac{r_m}{2}. \tag{7}$$

Using (3) and (6), we obtain that, for any $1 \leq i \leq m - i_m$,

$$\begin{aligned}\lambda_i(X+\Delta) - \lambda_m(X) \;&\geq\; \lambda_{m-i_m}(X+\Delta) - \lambda_m(X)\\ &> \lambda_{m-i_m}(X) - \lambda_m(X) - \frac{r_m}{2} \;\geq\; 2r_m - \frac{r_m}{2} \;=\; \frac{3r_m}{2}.\end{aligned} \tag{8}$$

Similarly, using (3) and (7), we obtain that, for any $m + j_m + 1 \leq i \leq n$,

$$\lambda_m(X) - \lambda_i(X+\Delta) \;>\; \frac{3r_m}{2}. \tag{9}$$

It follows from (8) and (9) that (4) holds. Finally, using Lemma 1, we obtain that, for $m - i_m + 1 \leq i \leq m + j_m$,

$$|\lambda_i(X+\Delta) - \lambda_m(X)| = |\lambda_i(X+\Delta) - \lambda_i(X)| \leq \|\Delta\| < \frac{r_m}{2},$$

and hence (5) holds. ∎

In this paper, we will make extensive use of the representation of a linear operator in terms of a contour integral involving its resolvent function. We now review a few basic facts below and refer the readers to [7] for further details.

For any $W \in \mathcal{S}^n$, the resolvent function $\lambda \mapsto (\lambda I - W)^{-1}$ is an (entrywise) analytic function on $\mathbb{C}\backslash\{\lambda_1(W), ..., \lambda_n(W)\}$. Thus, for any simple closed curve C not passing through any of the eigenvalues of W, one can define

$$\Pi_C(W) := \frac{1}{2\pi\iota} \oint_C (\lambda I - W)^{-1} d\lambda,$$

where ι denotes the imaginary unit, i.e., $\iota^2 = -1$. Let $W = \sum_{i=1}^n \lambda_i(W) u_i u_i^T$ be the eigenvalue decomposition of W. Then $(\lambda I - W)^{-1} = \sum_{i=1}^n \frac{1}{\lambda - \lambda_i(W)} u_i u_i^T$ and thus

$$\Pi_{_C}(W) = \frac{1}{2\pi\iota} \oint_C \sum_{i=1}^n \frac{1}{\lambda - \lambda_i(W)} u_i u_i^T d\lambda = \sum_{i=1}^n \left(\frac{1}{2\pi\iota} \oint_C \frac{1}{\lambda - \lambda_i(W)} d\lambda \right) u_i u_i^T = \sum_{i:\ \lambda_i(W) \in \mathrm{int}(C)} u_i u_i^T,$$

where $\mathrm{int}(C)$ denotes the interior of the simple closed curve C. Similarly, one can show that

$$W\Pi_{_C}(W) = \sum_{i:\ \lambda_i(W) \in \mathrm{int}(C)} W u_i u_i^T = \sum_{i:\ \lambda_i(W) \in \mathrm{int}(C)} \lambda_i(W) u_i u_i^T = \frac{1}{2\pi\iota} \oint_C \lambda(\lambda I - W)^{-1} d\lambda. \quad (10)$$

The following result provides an integral representation of the difference between $W\Pi_{_C}(W)$ and $(W + \Delta)\Pi_{_C}(W + \Delta)$.

Lemma 3. *Let W, $\Delta \in \mathcal{S}^n$ and C be a simple closed curve that does not pass through any eigenvalues of $W + \Delta$ and W. Then*

$$\begin{aligned}
&(W + \Delta)\Pi_{_C}(W + \Delta) - W\Pi_{_C}(W) \\
&= \frac{1}{2\pi\iota} \oint_C \lambda(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1} d\lambda + \frac{1}{2\pi\iota} \oint_C \lambda(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1}\Delta(\lambda I - W - \Delta)^{-1} d\lambda \quad (11) \\
&= \frac{1}{2\pi\iota} \oint_C \lambda(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1} d\lambda + \frac{1}{2\pi\iota} \oint_C \lambda(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1} d\lambda \\
&\quad + \frac{1}{2\pi\iota} \oint_C \lambda(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1}\Delta(\lambda I - W - \Delta)^{-1} d\lambda \quad (12)
\end{aligned}$$

Proof. First, notice that for any $\lambda \in C$, we have

$$\begin{aligned}
(\lambda I - W - \Delta)^{-1} - (\lambda I - W)^{-1} &= (\lambda I - W)^{-1} \left[\lambda I - W - (\lambda I - W - \Delta)\right] (\lambda I - W - \Delta)^{-1} \\
&= (\lambda I - W)^{-1}\Delta(\lambda I - W - \Delta)^{-1}. \quad (13)
\end{aligned}$$

Hence, we obtain that

$$(\lambda I - W - \Delta)^{-1} = (\lambda I - W)^{-1} + (\lambda I - W)^{-1}\Delta(\lambda I - W - \Delta)^{-1}. \quad (14)$$

Substituting this equality into the right-hand side of (13), we have

$$(\lambda I - W - \Delta)^{-1} - (\lambda I - W)^{-1} = (\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1} + (\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1}\Delta(\lambda I - W - \Delta)^{-1}.$$

Multiplying both sides of this relation by λ, integrating along C and using (10), we obtain (11). Substituting the equality (14) into (11), we further see that (12) holds. ■

Lemma 4. *Suppose $A \in \Re^{n\times n}$, $B \in \Re^{k\times n}$, $C \in \Re^{k\times k}$. Then*

$$\left\| \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \right\| \le \|A\| + \|C\| + 2\|B\|.$$

Proof. We observe that

$$\left\| \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \right\| = \max_{\|x\|^2+\|y\|^2\le 1} \left\| \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \max_{\|x\|^2+\|y\|^2\le 1} \sqrt{\|Ax+B^Ty\|^2 + \|Bx+Cy\|^2}$$
$$\le \max_{\|x\|^2+\|y\|^2\le 1} (\|Ax+B^Ty\| + \|Bx+Cy\|) \le \|A\| + \|C\| + 2\|B\|.$$

■

The following lemma can be readily obtained from [16, Page 230, Theorem 2.1]. A variant of this lemma has been used in [17] to establish (2).

Lemma 5. *Let $X \in \mathcal{S}^n$ and $X = \sum_{i=1}^n \lambda_i(X) u_i u_i^T$ be its eigenvalue decomposition. Let*

$$U_m := \begin{pmatrix} u_{m-i_m+1} & \cdots & u_{m+j_m} \end{pmatrix}, \qquad \widetilde{U}_m := \begin{pmatrix} u_1 & \cdots & u_{m-i_m} & u_{m+j_m+1} & \cdots & u_n \end{pmatrix}. \tag{15}$$

Then, for any $\|\Delta\| < r_m/2$, there exists a matrix $P \in \Re^{(n-i_m-j_m)\times(i_m+j_m)}$ with $\|P\| \le 2\|\Delta\|/r_m$ such that the image of V_1 and V_2 are invariant subspaces of $X+\Delta$, where V_1 and V_2 are given by

$$V_1 = (U_m + \widetilde{U}_m P)(I + P^T P)^{-\frac{1}{2}}, \qquad V_2 = (\widetilde{U}_m - U_m P^T)(I + PP^T)^{-\frac{1}{2}}. \tag{16}$$

3 Error bounds for first- and second-order approximations of eigenvalues

In this section, we establish error bounds for first- and second-order approximations for the mth eigenvalue of a symmetric matrix $X \in \mathcal{S}^n$ for any $1 \le m \le n$. Throughout this section, we assume that $X = \sum_{i=1}^n \lambda_i(X) u_i u_i^T$ is the eigenvalue decomposition of X, and that U_m and $\tilde{U}_m$ are defined as in (15). In addition, we define i_m, j_m and r_m as in Section 2.

3.1 Error bounds for first-order approximation of eigenvalues

The following proposition will be used subsequently to establish our main theorem of this subsection.

Proposition 1. *Let C_m be the circle centered at the origin with radius r_m, and let $W = X - \lambda_m(X)I$ and U_m be defined in (15). Then we have*

$$\|(W+\Delta)\Pi_{C_m}(W+\Delta) - U_m U_m^T \Delta U_m U_m^T\| \le \frac{2}{r_m}\|\Delta\|^2$$

whenever $\Delta \in \mathcal{S}^n$ and $\|\Delta\| < r_m/2$.

Proof. Suppose $\Delta \in \mathcal{S}^n$ such that $\|\Delta\| < r_m/2$. Notice that the eigenvalues of W and $W+\Delta$ are $\{\lambda_i(X) - \lambda_m(X) : \ i = 1, ..., n\}$ and $\{\lambda_i(X+\Delta) - \lambda_m(X) : \ i = 1, ..., n\}$, respectively. By the definition of i_m, j_m and r_m, we see that

$$\begin{aligned} &\lambda_i(W) = \lambda_i(X) - \lambda_m(X) = 0, && m - i_m + 1 \le i \le m + j_m, \\ &|\lambda_i(W)| = |\lambda_i(X) - \lambda_m(X)| \ge 2r_m, && i \le m - i_m \text{ or } i > m + j_m. \end{aligned} \tag{17}$$

Using (17) and Lemma 2, we observe that C_m does not go through any eigenvalue of W and $W+\Delta$, and moreover, the eigenvalues $\{\lambda_i(W)\}$ and $\{\lambda_i(W+\Delta)\}$ lie in the interior of C_m precisely when $m-i_m+1 \le i \le m+j_m$. Hence, we have

$$\begin{aligned}\frac{1}{2\pi\iota}\oint_{C_m} \lambda(\lambda I-W)^{-1}\Delta(\lambda I-W)^{-1}d\lambda &= \sum_{i,j}\left(\frac{1}{2\pi\iota}\oint_{C_m}\frac{\lambda\ u_iu_i^T\Delta u_ju_j^T}{[\lambda-\lambda_i(W)][\lambda-\lambda_j(W)]}d\lambda\right)\\ &= \sum_{(i,j)\in\mathcal{I}_m} u_iu_i^T\Delta u_ju_j^T = U_mU_m^T\Delta U_mU_m^T, \qquad (18)\end{aligned}$$

where $\mathcal{I}_m = \{(i,j):\ \lambda_i(W)=\lambda_j(W)=0\}$. In addition, it follows from (10) and (17) that

$$W\Pi_{C_m}(W) = \sum_{i:\ \lambda_i(W)\in \mathrm{int}(C_m)} \lambda_i(W)u_iu_i^T = \sum_{i=m-i_m+1}^{m+j_m}\lambda_i(W)u_iu_i^T = 0. \qquad (19)$$

In view of (17) and Lemma 2, we have that for any $\lambda\in C_m$,

$$\begin{aligned}\|(\lambda I-W)^{-1}\| &= \max_{1\le i\le n}\frac{1}{|\lambda-\lambda_i(W)|} = \frac{1}{|\lambda|} = \frac{1}{r_m}, &(20)\\ \|(\lambda I-W-\Delta)^{-1}\| &= \max_{1\le i\le n}\frac{1}{|\lambda-\lambda_i(W+\Delta)|} \le \max_{1\le i\le n}\frac{1}{||\lambda|-|\lambda_i(W+\Delta)||} \le \frac{2}{r_m}. &(21)\end{aligned}$$

Lemma 3 together with (18)–(21) yields

$$\begin{aligned}&\|(W+\Delta)\Pi_{C_m}(W+\Delta) - U_mU_m^T\Delta U_mU_m^T\|\\ &= \left\|(W+\Delta)\Pi_{C_m}(W+\Delta) - W\Pi_{C_m}(W) - \frac{1}{2\pi\iota}\oint_{C_m}\lambda(\lambda I-W)^{-1}\Delta(\lambda I-W)^{-1}d\lambda\right\|\\ &= \left\|\frac{1}{2\pi\iota}\oint_{C_m}\lambda(\lambda I-W)^{-1}\Delta(\lambda I-W)^{-1}\Delta(\lambda I-W-\Delta)^{-1}d\lambda\right\|\\ &\le \frac{1}{2\pi}\oint_{C_m}|\lambda|\|(\lambda I-W)^{-1}\|^2\|\Delta\|^2\|(\lambda I-W-\Delta)^{-1}\||d\lambda| \ \le\ \frac{2}{r_m}\|\Delta\|^2.\end{aligned}$$

This completes the proof. ■

We are now ready to establish the main theorem of this subsection.

Theorem 1. *For any $\Delta\in\mathcal{S}^n$ such that $\|\Delta\|<r_m/2$, we have*

$$|\lambda_m(X+\Delta)-\lambda_m(X)-\lambda_{i_m}(U_m^T\Delta U_m)| \le \frac{4}{r_m}\|\Delta\|^2, \qquad (22)$$

where U_m is defined in (15).

Proof. For simplicity of notation, let $W = X-\lambda_m(X)I$. We first observe that the polynomials $\Delta\mapsto\det(W+\Delta)$ and $\Delta\mapsto\det(U_m^T\Delta U_m)$ are not identically zero. Thus, the set

$$\Xi = \left\{\Delta\in\mathcal{S}^n:\ \|\Delta\|<\frac{r_m}{2},\ \det(W+\Delta)\ne 0,\ \det(U_m^T\Delta U_m)\ne 0\right\}$$

is dense in $\{\Delta \in \mathcal{S}^n : \|\Delta\| < r_m/2\}$. By the continuity of eigenvalues, it thus suffices to show that (22) holds on Ξ. We now assume that Δ is an arbitrary matrix in Ξ. Notice that $\lambda_i(W+\Delta)$ and $\lambda_j(U_m^T \Delta U_m)$ are nonzero for all $1 \le i \le n$ and $1 \le j \le i_m + j_m$. Let C_m be the circle centered at the origin with radius r_m. Recall from the proof of Proposition 1 that C_m does not go through any $\lambda_i(W+\Delta)$, and moreover, the eigenvalues $\{\lambda_i(W+\Delta)\}$ lie in the interior of C_m precisely when $m - i_m + 1 \le i \le m + j_m$. Define

$$R = (W+\Delta)\Pi_{C_m}(W+\Delta), \qquad S = U_m U_m^T \Delta U_m U_m^T.$$

We observe from (10) that R has exactly $i_m + j_m$ nonzero eigenvalues, which are

$$\{\lambda_i(W+\Delta) : \; m - i_m + 1 \le i \le m + j_m\}.$$

In addition, since $U_m^T U_m = 1$, it follows from [6, Theorem 1.3.20] that S and $U_m^T \Delta U_m$ share identical nonzero eigenvalues. Using this observation and the assumption $\Delta \in \Xi$, we conclude that S has exactly $i_m + j_m$ nonzero eigenvalues, which are

$$\{\lambda_i(U_m^T \Delta U_m) : \quad 1 \le i \le i_m + j_m\}.$$

Also, it follows from Proposition 1 and Lemma 1 that for all i,

$$|\lambda_i(R) - \lambda_i(S)| \;\le\; \|R - S\| \;=\; \|(W+\Delta)\Pi_{C_m}(W+\Delta) - U_m U_m^T \Delta U_m U_m^T\| \;\le\; \frac{2}{r_m}\|\Delta\|^2. \quad (23)$$

We next show that (22) holds for $\Delta \in \Xi$ by considering the following four cases.

Case 1. $\lambda_m(W+\Delta) > 0$ and $\lambda_{i_m}(U_m^T \Delta U_m) > 0$. In this case, one can observe that

$$\lambda_{i_m}(R) = \lambda_m(W+\Delta) \text{ and } \lambda_{i_m}(S) = \lambda_{i_m}(U_m^T \Delta U_m),$$

which together with (23) implies that

$$\begin{aligned} |\lambda_m(X+\Delta) - \lambda_m(X) - \lambda_{i_m}(U_m^T \Delta U_m)| &= |\lambda_m(W+\Delta) - \lambda_{i_m}(U_m^T \Delta U_m)| \\ &= |\lambda_{i_m}(R) - \lambda_{i_m}(S)| \;\le\; \frac{2}{r_m}\|\Delta\|^2. \end{aligned}$$

Case 2. $\lambda_m(W+\Delta) < 0$ and $\lambda_{i_m}(U_m^T \Delta U_m) < 0$. In this case, we can observe that

$$\lambda_{n-j_m}(R) = \lambda_m(W+\Delta) \;\text{ and }\; \lambda_{n-j_m}(S) = \lambda_{i_m}(U_m^T \Delta U_m).$$

Using these relations and (23), we obtain that

$$\begin{aligned} |\lambda_m(X+\Delta) - \lambda_m(X) - \lambda_{i_m}(U_m^T \Delta U_m)| &= |\lambda_m(W+\Delta) - \lambda_{i_m}(U_m^T \Delta U_m)| \\ &= |\lambda_{n-j_m}(R) - \lambda_{n-j_m}(S)| \;\le\; \frac{2}{r_m}\|\Delta\|^2. \end{aligned}$$

Case 3. $\lambda_m(W+\Delta) > 0$ and $\lambda_{i_m}(U_m^T \Delta U_m) < 0$. In this case, we have

$$\lambda_{i_m}(R) = \lambda_m(W+\Delta) > 0 \quad \text{and} \quad \lambda_{n-j_m}(S) = \lambda_{i_m}(U_m^T \Delta U_m) < 0.$$

Claim that $\lambda_{n-j_m}(R) \geq 0$ and $\lambda_{i_m}(S) \leq 0$. First, suppose to the contrary that $\lambda_{n-j_m}(R) < 0$. Then one must have

$$\lambda_1(R) \geq \cdots \geq \lambda_{i_m}(R) > 0 > \lambda_{n-j_m}(R) \geq \cdots \geq \lambda_n(R).$$

It implies that R has at least $i_m + j_m + 1$ nonzero eigenvalues, which contradicts with the fact that the number of nonzero eigenvalues of R is $i_m + j_m$. Similarly, we can show that $\lambda_{i_m}(S) \leq 0$. Using these facts and (23), we obtain

$$|\lambda_{i_m}(R)| = \lambda_{i_m}(R) \leq \lambda_{i_m}(R) - \lambda_{i_m}(S) = |\lambda_{i_m}(R) - \lambda_{i_m}(S)| \leq \frac{2}{r_m}\|\Delta\|^2,$$
$$|\lambda_{n-j_m}(S)| = -\lambda_{n-j_m}(S) \leq \lambda_{n-j_m}(R) - \lambda_{n-j_m}(S) = |\lambda_{n-j_m}(R) - \lambda_{n-j_m}(S)| \leq \frac{2}{r_m}\|\Delta\|^2.$$

Combining these two relations, we see that

$$\begin{aligned} |\lambda_m(X+\Delta) - \lambda_m(X) - \lambda_{i_m}(U_m^T \Delta U_m)| &= |\lambda_m(W+\Delta) - \lambda_{i_m}(U_m^T \Delta U_m)| \\ &= |\lambda_{i_m}(R) - \lambda_{n-j_m}(S)| \leq \frac{4}{r_m}\|\Delta\|^2. \end{aligned}$$

Case 4. $\lambda_m(W+\Delta) < 0$ and $\lambda_{i_m}(U_m^T \Delta U_m) > 0$. In this case, we see that

$$\lambda_{n-j_m}(R) = \lambda_m(W+\Delta) < 0 \quad \text{and} \quad \lambda_{i_m}(S) = \lambda_{i_m}(U_m^T \Delta U_m) > 0.$$

Using the similar argument as in Case 3, one can show that $\lambda_{i_m}(R) \leq 0$ and $\lambda_{n-j_m}(S) \geq 0$. By these inequalities and (23), we obtain that

$$|\lambda_{n-j_m}(R)| = -\lambda_{n-j_m}(R) \leq \lambda_{n-j_m}(S) - \lambda_{n-j_m}(R) = |\lambda_{n-j_m}(R) - \lambda_{n-j_m}(S)| \leq \frac{2}{r_m}\|\Delta\|^2,$$
$$|\lambda_{i_m}(S)| = \lambda_{i_m}(S) \leq \lambda_{i_m}(S) - \lambda_{i_m}(R) = |\lambda_{i_m}(R) - \lambda_{i_m}(S)| \leq \frac{2}{r_m}\|\Delta\|^2.$$

By virtue of these two relations, we further obtain that

$$\begin{aligned} |\lambda_m(X+\Delta) - \lambda_m(X) - \lambda_{i_m}(U_m^T \Delta U_m)| &= |\lambda_m(W+\Delta) - \lambda_{i_m}(U_m^T \Delta U_m)| \\ &= |\lambda_{n-j_m}(R) - \lambda_{i_m}(S)| \leq \frac{4}{r_m}\|\Delta\|^2. \end{aligned}$$

Combining the above four cases, we see that (22) holds for all $\Delta \in \Xi$. This together with the continuity of eigenvalues and the fact that Ξ is dense in $\{\Delta \in \mathcal{S}^n : \|\Delta\| < r_m/2\}$ leads to the conclusion of this theorem. ∎

As an immediate consequence, we obtain the first-order directional derivative of eigenvalues of real symmetric matrices that is established in [5].

Corollary 1. *For any X, $\Delta \in \mathcal{S}^n$, the first-order directional derivative $\lambda'_m(X;\Delta)$ defined in* (1) *is given by*

$$\lambda'_m(X;\Delta) = \lambda_{i_m}(U_m^T \Delta U_m),$$

where U_m is defined in (15).

3.2 Error bounds for second-order approximation for eigenvalues

The following proposition will be used subsequently to establish our main theorems of this subsection.

Proposition 2. *Let C_m be the circle centered at the origin with radius r_m, $W = X - \lambda_m(X)I$, U_m and $\widetilde{U}_m$ be defined in* (15), *and*

$$\tilde{\Lambda}_m := \mathrm{Diag}\left(-\lambda_1(W), \ldots, -\lambda_{m-i_m}(W), -\lambda_{m+j_m+1}(W), \ldots, -\lambda_n(W)\right). \tag{24}$$

Then we have

$$\|(W+\Delta)\Pi_{C_m}(W+\Delta) - U_m U_m^T \Delta U_m U_m^T - U_m U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T\| \le \frac{6}{r_m^2}\|\Delta\|^3 + \frac{8}{r_m^3}\|\Delta\|^4 \tag{25}$$

whenever $\Delta \in \mathcal{S}^n$ and $\|\Delta\| < r_m/2$.

Proof. Suppose $\Delta \in \mathcal{S}^n$ such that $\|\Delta\| < r_m/2$. Recall from the proof of Proposition 1 that C_m does not go through any eigenvalue of W and $W + \Delta$, and moreover, the eigenvalues $\{\lambda_i(W)\}$ and $\{\lambda_i(W+\Delta)\}$ lie in the interior of C_m precisely when $m - i_m + 1 \le i \le m + j_m$. Let

$$\mathcal{J}_m = \{(i,j,k):\ \lambda_m(X) = \lambda_j(X) = \lambda_k(X) \ne \lambda_i(X)\}.$$

It follows from (17) and (24) that

$$\|\tilde{\Lambda}_m^{-1}\| \ \le\ \frac{1}{2r_m}. \tag{26}$$

Further, we have

$$\begin{aligned}
&\frac{1}{2\pi\iota}\oint_{C_m} \lambda(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1} d\lambda \\
&= \sum_{i,j,k}\left(\frac{1}{2\pi\iota}\oint_{C_m} \frac{\lambda\ u_i u_i^T \Delta u_j u_j^T \Delta u_k u_k^T}{[\lambda - \lambda_i(W)]\ [\lambda - \lambda_j(W)]\ [\lambda - \lambda_k(W)]} d\lambda\right) \\
&= \sum_{(i,j,k)\in\mathcal{J}_m} \frac{u_i u_i^T \Delta u_j u_j^T \Delta u_k u_k^T}{-\lambda_i(W)} + \sum_{(j,i,k)\in\mathcal{J}_m} \frac{u_i u_i^T \Delta u_j u_j^T \Delta u_k u_k^T}{-\lambda_j(W)} + \sum_{(k,i,j)\in\mathcal{J}_m} \frac{u_i u_i^T \Delta u_j u_j^T \Delta u_k u_k^T}{-\lambda_k(W)} \\
&= \underbrace{\widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T \Delta U_m U_m^T}_{T_1} + \underbrace{U_m U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T}_{T_2} + \underbrace{U_m U_m^T \Delta U_m U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T}_{T_3}.
\end{aligned}$$

Using this relation along with Lemma 3 and (18)–(21), we obtain that

$$\begin{aligned}
&\|(W+\Delta)\Pi_{C_m}(W+\Delta) - U_m U_m^T \Delta U_m U_m^T - T_1 - T_2 - T_3\| \\
&\le \left\| \frac{1}{2\pi\iota} \oint_{C_m} \lambda(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1}\Delta(\lambda I - W)^{-1}\Delta(\lambda I - W - \Delta)^{-1} d\lambda \right\| \\
&\le \frac{1}{2\pi} \oint_{C_m} |\lambda| \|(\lambda I - W)^{-1}\|^3 \|\Delta\|^3 \|(\lambda I - W - \Delta)^{-1}\| |d\lambda| \le \frac{2}{r_m^2}\|\Delta\|^3. \qquad (27)
\end{aligned}$$

Let V_1, V_2 and P be defined in Lemma 5. In view of (16), (26) and the fact that $\|(I + PP^T)^{-1/2}\| \le 1$, we have

$$\begin{aligned}
\|V_1^T T_1 V_1\| &= \|V_1^T \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T \Delta U_m U_m^T V_1\| \le \|P^T \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T \Delta U_m\| \\
&\le \|P\| \|\Delta\|^2 \|\tilde{\Lambda}_m^{-1}\| \le \frac{1}{r_m^2}\|\Delta\|^3. \\
\|V_2^T T_1 V_2\| &= \|V_2^T \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T \Delta U_m U_m^T V_2\| \le \|\tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T \Delta U_m P^T\| \le \frac{1}{r_m^2}\|\Delta\|^3. \\
\|V_1^T T_1 V_2\| &= \|V_1^T \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T \Delta U_m U_m^T V_2\| \le \|P^T \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T \Delta U_m P^T\| \le \frac{2}{r_m^3}\|\Delta\|^4.
\end{aligned}$$

We can observe from (16) that the columns of V_1 and V_2 form an orthonormal basis. Using this fact, the above relations and Lemma 4, we obtain that

$$\|T_1\| = \left\| \begin{pmatrix} V_1^T T_1 V_1 & V_1^T T_1 V_2 \\ V_2^T T_1 V_1 & V_2^T T_1 V_2 \end{pmatrix} \right\| \le \|V_1^T T_1 V_1\| + \|V_2^T T_1 V_2\| + 2\|V_1^T T_1 V_2\| \le \frac{2}{r_m^2}\|\Delta\|^3 + \frac{4}{r_m^3}\|\Delta\|^4,$$

$$\|T_3\| = \|T_1^T\| = \|T_1\| \le \frac{2}{r_m^2}\|\Delta\|^3 + \frac{4}{r_m^3}\|\Delta\|^4.$$

Using these two inequalities and (27), we see that

$$\begin{aligned}
&\|(W+\Delta)\Pi_{C_m}(W+\Delta) - U_m U_m^T \Delta U_m U_m^T - U_m U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m U_m^T\| \\
&= \|(W+\Delta)\Pi_{C_m}(W+\Delta) - U_m U_m^T \Delta U_m U_m^T - T_2\| \\
&\le \|(W+\Delta)\Pi_{C_m}(W+\Delta) - U_m U_m^T \Delta U_m U_m^T - T_1 - T_2 - T_3\| + \|T_1\| + \|T_3\| \\
&\le \frac{6}{r_m^2}\|\Delta\|^3 + \frac{8}{r_m^3}\|\Delta\|^4,
\end{aligned}$$

which is just (25). This completes the proof. ■

We are now ready to establish our first main theorem of this subsection.

Theorem 2. *For any $\Delta \in \mathcal{S}^n$ such that $\|\Delta\| < r_m/2$, we have*

$$|\lambda_m(X+\Delta) - \lambda_m(X) - \lambda_{i_m}(U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m)| \le \frac{12}{r_m^2}\|\Delta\|^3 + \frac{16}{r_m^3}\|\Delta\|^4, \quad (28)$$

where U_m, $\tilde{\Lambda}_m$ and $\widetilde{U}_m$ are defined in (15) and (24) respectively.

Proof. For simplicity of notation, let $W = X - \lambda_m(X)I$. We first observe that the polynomials $\Delta \mapsto \det(W + \Delta)$ and $\Delta \mapsto \det(U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m)$ are not identically zero. Thus, the set

$$\Xi = \left\{ \Delta \in \mathcal{S}^n : \ \|\Delta\| < \frac{r_m}{2}, \ \det(W + \Delta) \neq 0, \ \det(U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m) \neq 0 \right\}$$

is dense in $\{\Delta \in \mathcal{S}^n : \|\Delta\| < r_m/2\}$. By the continuity of eigenvalues, it thus suffices to show that (28) holds on Ξ. We now assume that Δ is an arbitrary matrix in Ξ. Notice that $\lambda_i(W+\Delta)$ and $\lambda_j(U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m)$ are nonzero for all $1 \leq i \leq n$ and $1 \leq j \leq i_m + j_m$. Let C_m be the circle centered at the origin with radius r_m. Define

$$R = (W + \Delta)\Pi_{C_m}(W + \Delta), \qquad S = U_m(U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m)U_m^T.$$

We know from the proof of Theorem 1 that R has exactly $i_m + j_m$ nonzero eigenvalues, which are

$$\{\lambda_i(W + \Delta) : \ m - i_m + 1 \leq i \leq m + j_m\}.$$

In addition, by a similar argument as in the proof of Theorem 1, one can show that S has exactly $i_m + j_m$ nonzero eigenvalues, which are

$$\{\lambda_i(U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m) : \ 1 \leq i \leq i_m + j_m\}.$$

Also, it follows from Proposition 2 and Lemma 1 that for all i,

$$|\lambda_i(R) - \lambda_i(S)| \ \leq \ \|R - S\| \leq \frac{6}{r_m^2}\|\Delta\|^3 + \frac{8}{r_m^3}\|\Delta\|^4. \tag{29}$$

Proceeding similarly as in the proof of Theorem 1 by using (29) in place of (23) and replacing $U_m^T \Delta U_m$ by $U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m$, one can show that for any $\Delta \in \Xi$,

$$|\lambda_m(X + \Delta) - \lambda_m(X) - \lambda_{i_m}(U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m)| \leq \frac{12}{r_m^2}\|\Delta\|^3 + \frac{16}{r_m^3}\|\Delta\|^4.$$

This together with the continuity of eigenvalues and the fact that Ξ is dense in $\{\Delta \in \mathcal{S}^n : \|\Delta\| < r_m/2\}$ shows that (28) holds for any $\Delta \in \mathcal{S}^n$ with $\|\Delta\| < r_m/2$. This completes the proof. ∎

Before stating the next theorem, we introduce some notations. For each $\Delta \in \mathcal{S}^n$, consider the $(i_m + j_m) \times (i_m + j_m)$ matrix $U_m^T \Delta U_m$. Let $U_m^T \Delta U_m = \sum_{i=1}^{i_m+j_m} \lambda_i(U_m^T \Delta U_m)\bar{u}_i \bar{u}_i^T$ be an eigenvalue decomposition. We define the integers $\bar{i}_m$ and $\bar{j}_m$ as the number of eigenvalues of $U_m^T \Delta U_m$ ranking before i_m that equal $\lambda_{i_m}(U_m^T \Delta U_m)$ and the number of eigenvalues ranking (strictly) after i_m that equal $\lambda_{i_m}(U_m^T \Delta U_m)$, respectively. We then define

$$\bar{U}_m = \begin{pmatrix} \bar{u}_{i_m - \bar{i}_m + 1} & \cdots & \bar{u}_{i_m + \bar{j}_m} \end{pmatrix}, \tag{30}$$

and

$$\bar{r}_m := \frac{1}{2}\min\{\lambda_{i_m - \bar{i}_m}(U_m^T \Delta U_m) - \lambda_{i_m}(U_m^T \Delta U_m), \lambda_{i_m}(U_m^T \Delta U_m) - \lambda_{i_m + \bar{j}_m + 1}(U_m^T \Delta U_m)\}.$$

It is easy to see that $\bar{r}_m > 0$. We are now ready to establish a theorem about the error bound along a fixed direction of perturbation.

Theorem 3. *Let $\Delta \in \mathcal{S}^n$. For any $0 < t < \min\left\{1, \frac{r_m \bar{r}_m}{\|\Delta\|^2}, \frac{r_m}{2\|\Delta\|}\right\}$, we have*

$$|\lambda_m(X + t\Delta) - \lambda_m(X) - t\lambda_{i_m}(U_m^T \Delta U_m) - t^2 \lambda_{\bar{i}_m}(\bar{U}_m^T U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m \bar{U}_m)| \leq \frac{12t^3}{r_m^2}\|\Delta\|^3 + \frac{16t^4}{r_m^3}\|\Delta\|^4 + \frac{t^3}{r_m^2 \bar{r}_m}\|\Delta\|^4, \tag{31}$$

where U_m, $\tilde{\Lambda}_m$, $\widetilde{U}_m$ and $\bar{U}_m$ are defined in (15), (24) *and* (30)*, respectively.*

Proof. Fix any $0 < t < \min\left\{1, \frac{r_m \bar{r}_m}{\|\Delta\|^2}, \frac{r_m}{2\|\Delta\|}\right\}$. Since $t\|\Delta\| < \frac{r_m}{2}$, we see from Theorem 2 that

$$|\lambda_m(X + t\Delta) - \lambda_m(X) - t\lambda_{i_m}(U_m^T \Delta U_m + tU_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m)| \leq \frac{12t^3}{r_m^2}\|\Delta\|^3 + \frac{16t^4}{r_m^3}\|\Delta\|^4. \tag{32}$$

In addition, using (26) and the fact that $t\|\Delta\|^2 < r_m \bar{r}_m$, we obtain

$$\|tU_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m\| \leq t\|\Delta\|^2 \|\tilde{\Lambda}_m^{-1}\| < \frac{\bar{r}_m}{2}.$$

Hence, by specializing X and Δ in Theorem 1 to $U_m^T \Delta U_m$ and $tU_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m$ respectively, we have

$$\begin{aligned} &|\lambda_{i_m}(U_m^T \Delta U_m + tU_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m) - \lambda_{i_m}(U_m^T \Delta U_m) - t\lambda_{\bar{i}_m}(\bar{U}_m^T U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m \bar{U}_m)| \\ &\leq \frac{4}{\bar{r}_m}\|tU_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m\|^2 \leq \frac{t^2}{r_m^2 \bar{r}_m}\|\Delta\|^4, \end{aligned} \tag{33}$$

where we made use of (26) in the last inequality. Adding (32) and (33) and using the triangle inequality, we obtain (31). This completes the proof. ∎

As a byproduct, the second-order directional derivative of eigenvalues of real symmetric matrices that is the main result established in [17] directly follows by combining Corollary 1 with Theorem 3.

Corollary 2. *For any X, $\Delta \in \mathcal{S}^n$, the second-order directional derivative $\lambda_m''(X; \Delta)$ defined in* (2) *is given by*

$$\lambda_m''(X; \Delta) = 2\lambda_{\bar{i}_m}(\bar{U}_m^T U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m \bar{U}_m),$$

where U_m, $\tilde{\Lambda}_m$, $\widetilde{U}_m$ and $\bar{U}_m$ are defined in (15), (24) *and* (30)*, respectively.*

4 Error bounds for first- and second-order approximations of singular values

In this section, we study error bounds for first- and second-order approximations for the mth singular value of a matrix $Z \in \Re^{p \times q}$ for any $1 \leq m \leq k := \min\{p, q\}$. We will make use of the fact that the singular values of Z correspond to some eigenvalues of the matrix

$$X := \begin{pmatrix} 0 & Z \\ Z^T & 0 \end{pmatrix} \in \Re^{(p+q) \times (p+q)} \tag{34}$$

(see, for example, [16, Page 32, Theorem 4.2]). Indeed, we denote the singular values of Z by

$$\sigma_1(Z) \geq \sigma_2(Z) \geq \cdots \geq \sigma_k(Z) \geq 0.$$

Let $Z = \sum_{i=1}^k \sigma_i(Z) g_i h_i^T$ be a singular value decomposition of Z, where $\{g_1, \ldots, g_k\}$ and $\{h_1, \ldots, h_k\}$ are orthonormal vectors, respectively. Let

$$u_i := \frac{1}{\sqrt{2}} \begin{pmatrix} g_i \\ h_i \end{pmatrix}, \quad u_{p+q+1-i} := \frac{1}{\sqrt{2}} \begin{pmatrix} g_i \\ -h_i \end{pmatrix}, \quad i = 1, ..., k,$$

and $\{u_i\}_{i=k+1}^{p+q-k}$ be the orthonormal vectors perpendicular to $\{u_i\}_{i=1}^k$ and $\{u_{p+q+1-i}\}_{i=1}^k$. Then, the eigenvalues $\{\lambda_i(X)\}_{i=1}^{p+q}$ of X are

$$\sigma_1(Z), \ldots, \sigma_q(Z), 0, \ldots, 0, -\sigma_q(Z), \ldots, -\sigma_1(Z), \tag{35}$$

and the corresponding orthonormal eigenvectors are $\{u_i\}_{i=1}^{p+q}$.

For any $1 \leq m \leq k$, let i_m, j_m and r_m be defined as in Section 2 for the above X. In view of (3) and (35), we have

$$r_m = \begin{cases} \frac{1}{2} \min\{\sigma_{m-i_m}(Z) - \sigma_m(Z), \sigma_m(Z) - \sigma_{m+j_m+1}(Z)\}, & \text{if } \sigma_m(Z) > 0, \\ \frac{1}{2} \sigma_{m-i_m}(Z), & \text{if } \sigma_m(Z) = 0. \end{cases} \tag{36}$$

Also, let U_m, $\widetilde{U}_m$ and $\tilde{\Lambda}_m$ be defined as in (15) and (24), respectively. Finally, for any perturbation $E \in \Re^{p \times q}$, define

$$\Delta := \begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix}. \tag{37}$$

We are now ready to state our main result about error bounds for first- and second-order approximation of singular values of Z.

Theorem 4. *Let $Z \in \Re^{p \times q}$, $1 \leq m \leq k := \min\{p, q\}$, r_m and Δ be defined in (36) and (37), respectively.*

(i) For any $E \in \Re^{p \times q}$ with $\|E\| < r_m/2$, we have

$$|\sigma_m(Z + E) - \sigma_m(Z) - \lambda_{i_m}(U_m^T \Delta U_m)| \leq \frac{4}{r_m} \|E\|^2.$$

(ii) For any $E \in \Re^{p \times q}$ with $\|E\| < r_m/2$, we have

$$|\sigma_m(Z + E) - \sigma_m(Z) - \lambda_{i_m}(U_m^T \Delta U_m + U_m^T \Delta \widetilde{U}_m \tilde{\Lambda}_m^{-1} \widetilde{U}_m^T \Delta U_m)| \leq \frac{12}{r_m^2} \|E\|^3 + \frac{16}{r_m^3} \|E\|^4$$

Proof. In view of (34) and (37), we observe that

$$\lambda_m(X + \Delta) = \sigma_m(Z + E), \qquad \lambda_m(X) = \sigma_m(Z).$$

Moreover,

$$\|E\| \leq \sup_{\|x\|^2 + \|y\|^2 \leq 1} \sqrt{\|E^T x\|^2 + \|Ey\|^2} = \|\Delta\| \leq \sup_{\|x\|^2 + \|y\|^2 \leq 1} \sqrt{\|E^T\|^2 \|x\|^2 + \|E\|^2 \|y\|^2} = \|E\|,$$

which yields $\|E\| = \|\Delta\|$. The conclusions of this theorem then immediately follow from Theorems 1 and 2. ∎

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