

## Robust portfolio selection based on a joint ellipsoidal uncertainty set

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'Separable' uncertainty sets have been widely used in robust portfolio selection models (e.g. see [E. Erdoğan, D. Goldfarb, and G. Iyengar, *Robust portfolio management*, manuscript, Department of Industrial Engineering and Operations Research, Columbia University, New York, 2004; D. Goldfarb and G. Iyengar, *Robust portfolio selection problems*, Math. Oper. Res. 28 (2003), pp. 1–38; R.H. Tütüncü and M. Koenig, *Robust asset allocation*, Ann. Oper. Res. 132 (2004), pp. 157–187]). For these uncertainty sets, each type of uncertain parameter (e.g. mean and covariance) has its own uncertainty set. As addressed in [Z. Lu, *A new cone programming approach for robust portfolio selection*, Tech. Rep., Department of Mathematics, Simon Fraser University, Burnaby, BC, 2006; Z. Lu, *A computational study on robust portfolio selection based on a joint ellipsoidal uncertainty set*, Math. Program. (2009), DOI: 10.1007/510107-009-0271-z], these 'separable' uncertainty sets typically share two common properties: (1) their actual confidence level, namely, the probability of uncertain parameters falling within the uncertainty set, is unknown, and it can be much higher than the desired one; and (2) they are fully or partially box-type. The associated consequences are that the resulting robust portfolios can be too conservative, and moreover, they are usually highly non-diversified, as observed in the computational experiments conducted in [Z. Lu, *A new cone programming approach for robust portfolio selection*, Tech. Rep., Department of Mathematics, Simon Fraser University, Burnaby, BC, 2006; Z. Lu, *A computational study on robust portfolio selection based on a joint ellipsoidal uncertainty set*, Math. Program. (2009), DOI: 10.1007/510107-009-0271-z; R.H. Tütüncü and M. Koenig, *Robust asset allocation*, Ann. Oper. Res. 132 (2004), pp. 157–187]. To combat these drawbacks, we consider a factor model for random asset returns. For this model, we introduce a 'joint' ellipsoidal uncertainty set for the model parameters and show that it can be constructed as a confidence region associated with a statistical procedure applied to estimate the model parameters. We further show that the robust maximum risk-adjusted return (RMRAR) problem with this uncertainty set can be reformulated and solved as a cone programming problem. The computational results reported in [Z. Lu, *A new cone programming approach for robust portfolio selection*, Tech. Rep., Department of Mathematics, Simon Fraser University, Burnaby, BC, 2006; Z. Lu, *A computational study on robust portfolio selection based on a joint ellipsoidal uncertainty set*, Math. Program. (2009), DOI: 10.1007/510107-009-0271-z] demonstrate that the robust portfolio determined by the RMRAR model with our 'joint' uncertainty set outperforms that with Goldfarb and Iyengar's 'separable' uncertainty set proposed in the seminal paper [D. Goldfarb and G. Iyengar, *Robust portfolio selection problems*, Math. Oper. Res. 28 (2003), pp. 1–38] in terms of wealth growth rate and transaction cost; moreover, our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is surprisingly highly non-diversified.

**Keywords:** robust optimization; mean-variance portfolio selection; maximum risk-adjusted return portfolio selection; cone programming; linear regression

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## 1. Introduction

The portfolio selection problem is concerned with determining a portfolio such that the ‘return’ and ‘risk’ of the portfolio have a favourable trade-off. The first mathematical model for the portfolio selection problem was developed by Markowitz [23] five decades ago, in which an optimal or efficient portfolio can be identified by solving a convex quadratic program. In his model, the ‘return’ and ‘risk’ of a portfolio are measured by the mean and variance, respectively, of the random portfolio return. Thus, the Markowitz portfolio model is also widely referred to as the mean-variance model.

Despite the theoretical elegance and importance of the mean-variance model, it continues to encounter skepticism among investment practitioners. One of the main reasons is that the optimal portfolios determined by the mean-variance model are often sensitive to perturbations in the parameters of the problem (e.g. expected returns and the covariance matrix), and thus lead to large turnover ratios with periodic adjustments of the problem parameters; see, for example, Michaud [24]. Various aspects of this phenomenon have also been extensively studied in the literature, for example, see [7–9,11].

As a recently emerging modelling tool, robust optimization can incorporate the perturbations in the parameters of the problems into the decision-making process. Generally speaking, robust optimization aims to find solutions to the given optimization problems with uncertain problem parameters that will achieve good objective values for all or most of the realizations of the uncertain problem parameters. For details, see [1–3,12,13]. Recently, robust optimization has been applied to model portfolio selection problems in order to combat the sensitivity of optimal portfolios to statistical errors in the estimates of problem parameters. For example, Goldfarb and Iyengar [16] considered a factor model for random portfolio returns, and proposed some statistical procedures for constructing uncertainty sets for the model parameters. For these uncertainty sets, they showed that robust portfolio selection problems can be reformulated as second-order cone programs. Subsequently, Erdoğan *et al.* [15] extended this method to robust index tracking and active portfolio management problems. Alternatively, Tütüncü and Koenig [28] (see also Halldórsson and Tütüncü [17]) considered a box-type uncertainty structure for the mean and covariance matrix of the assets returns. For this uncertainty structure, they showed that the robust portfolio selection problems can be formulated and solved as smooth saddle-point problems that involve semidefinite constraints. In addition, for finite uncertainty sets, Ben-Tal *et al.* [4] studied the robust formulations of multi-stage portfolio selection problems. Also, El Ghaoui *et al.* [14] considered the robust value-at-risk (VaR) problems given the partial information on the distribution of the returns, and they showed that these problems can be cast as semidefinite programs. Zhu and Fukushima [30] showed that the robust conditional value-at-risk problems can be reformulated as linear programs or second-order cone programs for some simple uncertainty structures of the distributions of the returns. Recently, DeMiguel and Nogales [10] proposed a novel approach for portfolio selection by minimizing certain robust estimators of portfolio risk. In their method, robust estimation and portfolio optimization are performed by solving a single nonlinear program.

The structure of the uncertainty set is an important ingredient in formulating and solving robust portfolio selection problems. The ‘separable’ uncertainty sets have been commonly used in the literature. For example, Tütüncü and Koenig [28] (see also Halldórsson and Tütüncü [17]) proposed the box-type uncertainty sets  $\mathcal{S}_m = \{\mu : \mu^L \leq \mu \leq \mu^U\}$  and  $\mathcal{S}_v = \{\Sigma : \Sigma \succeq 0, \Sigma^L \leq \Sigma \leq \Sigma^U\}$  for the mean  $\mu$  and covariance  $\Sigma$  of the asset return vector  $r$ , respectively. Here,  $A \succeq 0$  (resp.  $\succ 0$ ) denotes that the matrix  $A$  is symmetric and positive semidefinite (resp. definite). In the seminal paper [16], Goldfarb and Iyengar studied a factor model for the asset return vector  $r$  in the form of

$$r = \mu + V^T f + \epsilon,$$

where  $\mu$  is the mean return vector,  $f$  is the random factor return vector,  $V$  is the factor loading matrix and  $\epsilon$  is the residual return vector (see Section 2 for details). They also proposed some uncertainty sets  $\mathcal{S}_m$  and  $\mathcal{S}_v$  for  $\mu$  and  $V$ , respectively; in particular,  $\mathcal{S}_m$  is a box and  $\mathcal{S}_v$ , a Cartesian product of a bunch of ellipsoids. It shall be mentioned that all the ‘separable’ uncertainty sets above share two common properties: (1) viewed as a joint uncertainty set, the actual confidence level of  $\mathcal{S}_m \times \mathcal{S}_v$  is unknown, although  $\mathcal{S}_m$  and/or  $\mathcal{S}_v$  may have known confidence levels individually; and (2)  $\mathcal{S}_m \times \mathcal{S}_v$  is fully or partially box-type. The associated consequences are that the resulting robust portfolio can be too conservative since the actual confidence level of  $\mathcal{S}_m \times \mathcal{S}_v$  can be much higher than the desired one, or equivalently, the uncertainty set  $\mathcal{S}_m \times \mathcal{S}_v$  can be too noisy; and moreover, it is highly non-diversified as observed in the computational experiments conducted in [21,22,28]. These drawbacks will be further addressed in detail in Section 2.

In this article, we also consider the same factor model for asset returns as studied in [16]. To combat the aforementioned drawbacks, we propose a ‘joint’ ellipsoidal uncertainty set for the model parameters  $(\mu, V)$ , and show that it can be constructed as a confidence region associated with a statistical procedure applied to estimate  $(\mu, V)$  for any desired confidence level. We further show that for this uncertainty set, the robust maximum risk-adjusted return (RMRAR) problem can be reformulated and solved as a cone programming problem. The computational results reported in [21,22] demonstrate that the robust portfolio determined by the RMRAR model with our ‘joint’ uncertainty set outperforms that with Goldfarb and Iyengar’s ‘separable’ uncertainty set proposed in the seminal paper [16] in terms of wealth growth rate and transaction cost; moreover, our robust portfolio is fairly diversified, but Goldfarb and Iyengar’s is surprisingly highly non-diversified.

The rest of this article is organized as follows. In Section 2, we describe the factor model for asset returns that was studied in Goldfarb and Iyengar [16], and briefly review the statistical procedure proposed in [16] for constructing a ‘separable’ uncertainty set of the model parameters. The associated drawbacks of this uncertainty set are also addressed. In Section 3, we introduce a ‘joint’ uncertainty set for the model parameters, and propose a statistical procedure for constructing it for any desired confidence level. Several robust portfolio selection problems for this uncertainty set are also discussed. In Section 4, we show that for our ‘joint’ uncertainty set, the RMRAR problem can be reformulated and solved as a cone programming problem. Finally, we give some concluding remarks in Section 5.

## 2. Factor model and separable uncertainty sets

In this section, we first describe the factor model for asset returns that was studied in Goldfarb and Iyengar [16]. Then, we briefly review the statistical procedure proposed in [16] for constructing a ‘separable’ uncertainty set for the model parameters. The associated drawbacks of this uncertainty set are also addressed.

The following factor model for asset returns was studied in [16]. Suppose that a discrete-time market has  $n$  traded assets. The vector of asset returns over a single market period is denoted by  $r \in \mathfrak{R}^n$ . The returns on the assets in different market periods are assumed to be independent. The single period return  $r$  is assumed to be a random vector given by

$$r = \mu + V^T f + \epsilon, \tag{1}$$

where  $\mu \in \mathfrak{R}^n$  is the vector of mean returns,  $f \sim \mathcal{N}(0, F) \in \mathfrak{R}^m$  denotes the returns of the  $m$  factors driving the market,  $V \in \mathfrak{R}^{m \times n}$  denotes the factor loading matrix of the  $n$  assets and  $\epsilon \sim \mathcal{N}(0, D) \in \mathfrak{R}^n$  is the vector of residual returns. Here  $x \sim \mathcal{N}(\mu, \Sigma)$  denotes that  $x$  is a multivariate normal random variable with mean  $\mu$  and covariance  $\Sigma$ . Further, it is assumed that  $D$  is a positive semidefinite diagonal matrix, and the residual return vector  $\epsilon$  is independent of the factor return vector  $f$ . Thus, it can be seen from the above assumption that  $r \sim \mathcal{N}(\mu, V^T F V + D)$ .

Goldfarb and Iyengar [16] also studied a robust model for (1). In their model, the mean return vector  $\mu$  is assumed to lie in the uncertainty set  $\mathcal{S}_m$  given by

$$\mathcal{S}_m = \{\mu : \mu = \mu_0 + \xi, |\xi_i| \leq \gamma_i, i = 1, \dots, n\}, \quad (2)$$

and the factor loading matrix  $V$  is assumed to belong to the uncertainty set  $\mathcal{S}_v$  given by

$$\mathcal{S}_v = \{V : V = V_0 + W, \|W_i\|_G \leq \rho_i, i = 1, \dots, n\}, \quad (3)$$

where  $W_i$  is the  $i$ th column of  $W$  and  $\|w\|_G = \sqrt{w^T G w}$  denotes the elliptic norm of  $w$  with respect to a symmetric, positive definite matrix  $G$ .

Two analogous statistical procedures were proposed in [16] for constructing the above uncertainty sets  $\mathcal{S}_m$  and  $\mathcal{S}_v$ . As observed in our computational experiments, the behaviour of these uncertainty sets is almost identical. Therefore, we only briefly review their first statistical procedure as follows. Suppose the market data consists of asset returns  $\{r^t : t = 1, \dots, p\}$  and factor returns  $\{f^t : t = 1, \dots, p\}$  for  $p$  trading periods. Then the linear model (1) implies that

$$r_i^t = \mu_i + \sum_{j=1}^m V_{ji} f_j^t + \epsilon_i^t, \quad i = 1, \dots, n, t = 1, \dots, p. \quad (4)$$

As in the typical linear regression analysis, it is assumed that  $\{\epsilon_i^t : i = 1, \dots, n, t = 1, \dots, p\}$  are all independent normal random variables and  $\epsilon_i^t \sim \mathcal{N}(0, \sigma_i^2)$  for all  $t = 1, \dots, p$ . Now, let  $B = (f^1, f^2, \dots, f^p) \in \mathfrak{R}^{m \times p}$  denote the matrix of factor returns, and let  $e \in \mathfrak{R}^p$  denote an all-one vector. Further, define

$$y_i = (r_i^1, r_i^2, \dots, r_i^p)^T, \quad A = (e B^T), \quad x_i = (\mu_i, V_{1i}, V_{2i}, \dots, V_{mi})^T, \quad \epsilon_i = (\epsilon_i^1, \dots, \epsilon_i^p)^T \quad (5)$$

for  $i = 1, \dots, n$ . Then (4) can be rewritten as

$$y_i = A x_i + \epsilon_i, \quad \forall i = 1, \dots, n. \quad (6)$$

Since usually  $p \gg m$  in practice, it was assumed in [16] that the matrix  $A$  has full column rank. As a result, it follows from (6) that the least-squares estimate  $\bar{x}_i$  of the true parameter  $x_i$  is given by

$$\bar{x}_i = (A^T A)^{-1} A^T y_i. \quad (7)$$

The following result has played a crucial role in constructing the uncertainty sets  $\mathcal{S}_m$  and  $\mathcal{S}_v$  in [16]. It will also be used to build a ‘joint’ ellipsoidal uncertainty set in Section 3.

**THEOREM 2.1** *Let  $s_i^2$  be the unbiased estimate of  $\sigma_i^2$  given by*

$$s_i^2 = \frac{\|y_i - A \bar{x}_i\|^2}{p - m - 1} \quad (8)$$

for  $i = 1, \dots, n$ . Then the random variables

$$\mathcal{Y}^i = \frac{1}{(m+1)s_i^2} (\bar{x}_i - x_i)^T A^T A (\bar{x}_i - x_i), \quad i = 1, \dots, n, \quad (9)$$

are distributed according to the  $F$ -distribution with  $m+1$  degrees of freedom in the numerator and  $p-m-1$  degrees of freedom in the denominator. Moreover,  $\{\mathcal{Y}^i\}_{i=1}^n$  are independent.

*Proof* The first statement was shown in [16, p. 16]. We now prove the second statement. In view of (6) and (7), we obtain that

$$\bar{x}_i - x_i = (A^T A)^{-1} A^T \epsilon_i, \quad y_i - A\bar{x}_i = [I - A(A^T A)^{-1} A^T] \epsilon_i.$$

Using these relations and the assumption that  $\{\epsilon_i: i = 1, \dots, n\}$  are independent, we conclude that  $\{\mathcal{Y}^i\}_{i=1}^n$  are independent. ■

Let  $\mathcal{F}_J$  denote the cumulative distribution function of the  $F$ -distribution with  $J$  degrees of freedom in the numerator and  $p - m - 1$  degrees of freedom in the denominator. Given any  $\tilde{\omega} \in (0, 1)$ , let  $c_J(\tilde{\omega})$  be its  $\tilde{\omega}$ -critical value, that is,  $\mathcal{F}_J(c_J(\tilde{\omega})) = \tilde{\omega}$ . Also, we let

$$\begin{aligned} \mathcal{S}^i(\tilde{\omega}) &= \{x_i : (\bar{x}_i - x_i)^T A^T A (\bar{x}_i - x_i) \leq (m + 1)c_{m+1}(\tilde{\omega})s_i^2\}, \quad i = 1, \dots, n, \\ \mathcal{S}(\tilde{\omega}) &= \mathcal{S}^1(\tilde{\omega}) \times \mathcal{S}^2(\tilde{\omega}) \times \dots \times \mathcal{S}^n(\tilde{\omega}). \end{aligned} \tag{10}$$

Using Theorem 2.1, Goldfarb and Iyengar showed that  $\mathcal{S}(\tilde{\omega})$  is an  $\tilde{\omega}^n$ -confidence set of  $(\mu, V)$ . Let  $\mathcal{S}_m(\tilde{\omega})$  and  $\mathcal{S}_v(\tilde{\omega})$  denote the projection of  $\mathcal{S}(\tilde{\omega})$  along  $\mu$  and  $V$ , respectively. Their explicit expressions can be found in [16, Section 5], and they are in the form of (2)–(3). Goldfarb and Iyengar [16] set  $\mathcal{S}_m := \mathcal{S}_m(\tilde{\omega})$  and  $\mathcal{S}_v := \mathcal{S}_v(\tilde{\omega})$ , and viewed them as the  $\tilde{\omega}^n$ -confidence sets of  $\mu$  and  $V$ , respectively. However, we immediately observe that

$$P(\mu \in \mathcal{S}_m(\tilde{\omega})) \geq P((\mu, V) \in \mathcal{S}(\tilde{\omega})) = \tilde{\omega}^n,$$

and similarly,  $P(V \in \mathcal{S}_v(\tilde{\omega})) \geq \tilde{\omega}^n$ . Hence,  $\mathcal{S}_m$  and  $\mathcal{S}_v$  have at least  $\tilde{\omega}^n$ -confidence levels, but their actual confidence levels are unknown and can be much higher than  $\tilde{\omega}^n$ . Further, in view of the relation  $\mathcal{S}(\tilde{\omega}) \subseteq \mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$ , one has

$$P((\mu, V) \in \mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})) \geq P((\mu, V) \in \mathcal{S}(\tilde{\omega})) = \tilde{\omega}^n.$$

Thus  $\mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$ , as a joint uncertainty set of  $(\mu, V)$ , has at least a  $\tilde{\omega}^n$ -confidence level. However, its actual confidence level is unknown and can be much higher than the desired one, that is,  $\tilde{\omega}^n$ . One immediate consequence is that the robust portfolio selection models based on such uncertainty sets  $\mathcal{S}_m(\tilde{\omega})$  and  $\mathcal{S}_v(\tilde{\omega})$  can be too conservative. In addition, we observe from computational experiments that the resulting robust portfolios are highly non-diversified, in other words, they concentrate only on a few assets. One possible interpretation of this phenomenon is that  $\mathcal{S}_m(\tilde{\omega})$  has a box-type structure. To combat these drawbacks, we introduce a ‘joint’ ellipsoidal uncertainty set for  $(\mu, V)$  in Section 3, and show that it can be constructed by a statistical approach.

Before ending this section, we shall remark that in Goldfarb and Iyengar’s robust factor model, some uncertainty structures were also proposed for the parameters  $F$  and  $D$  that are the covariances of factor and residual returns. Nevertheless, they are often assumed to be fixed in practical computations, and can be estimated by some standard statistical approaches [16, Section 7]. For brevity of presentation, we assume that  $F$  and  $D$  are fixed throughout the rest of the article. But we shall mention that the results of this article can be extended to the case where  $F$  and  $D$  have the same uncertainty structures as described in [16].

### 3. Joint uncertainty set and robust portfolio selection models

In this section, we consider the same factor model for asset returns as described in Section 2. In particular, we first introduce a ‘joint’ ellipsoidal uncertainty set for the model parameters  $(\mu, V)$ .

Then we propose a statistical procedure for constructing it. Finally, we discuss several robust portfolio selection models for this uncertainty set.

Throughout this section, we assume that all notations are given in Section 2, unless explicitly defined otherwise.

Recall from Section 2 that the ‘separable’ uncertainty set  $\mathcal{S}_\mu(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$  of  $(\mu, V)$  proposed in [16] has several drawbacks. To overcome these drawbacks, we consider a ‘joint’ ellipsoidal uncertainty set of  $(\mu, V)$  with a  $\omega$ -confidence level in the form of

$$\mathcal{S}_{\mu,v}(\omega) = \left\{ (\tilde{\mu}, \tilde{V}) \in \mathfrak{R}^n \times \mathfrak{R}^{m \times n} : \sum_{i=1}^n \frac{(\tilde{x}_i - \bar{x}_i)^T (A^T A) (\tilde{x}_i - \bar{x}_i)}{s_i^2} \leq (m+1)\tilde{c}(\omega) \right\} \quad (11)$$

for some  $\tilde{c}(\omega)$ , where  $\tilde{x}_i = (\tilde{\mu}_i, \tilde{V}_{1i}, \tilde{V}_{2i}, \dots, \tilde{V}_{mi})^T$  for  $i=1, \dots, n$ . Using the definition of  $\{\mathcal{Y}^i\}_{i=1}^n$  (see (9)), we can observe that the following property holds for  $\mathcal{S}_{\mu,v}(\omega)$ .

**PROPOSITION 3.1**  $\mathcal{S}_{\mu,v}(\omega)$  is an  $\omega$ -confidence uncertainty set of  $(\mu, V)$  for some  $\tilde{c}(\omega)$  if and only if  $P(\sum_{i=1}^n \mathcal{Y}^i \leq \tilde{c}(\omega)) = \omega$ , that is,  $\tilde{c}(\omega)$  is the  $\omega$ -critical value of  $\sum_{i=1}^n \mathcal{Y}^i$ .

We are now ready to propose a statistical procedure for constructing the aforementioned ‘joint’ ellipsoidal uncertainty set for the parameters  $(\mu, V)$ . First, we know from Theorem 2.1 that the random variables  $\{\mathcal{Y}^i\}_{i=1}^n$  have the  $F$ -distribution with  $m+1$  degrees of freedom in the numerator and  $p-m-1$  degrees of freedom in the denominator. It follows from a standard statistical result (e.g. see [26]) that their mean and standard deviation are

$$\mu_F = \frac{p-m-1}{p-m-3}, \quad \sigma_F = \sqrt{\frac{2(p-m-1)^2(p-2)}{(m+1)(p-m-3)^2(p-m-5)}}, \quad (12)$$

respectively, provided that  $p > m+5$ , which often holds in practice. In view of Theorem 2.1, we also know that  $\{\mathcal{Y}^i\}_{i=1}^n$  are i.i.d.. Using this fact and the central limit theorem (e.g. see [20]), we conclude that the distribution of the random variable

$$\mathcal{Z}_n = \frac{\sum_{i=1}^n \mathcal{Y}^i - n\mu_F}{\sigma_F \sqrt{n}}$$

converges towards the standard normal distribution  $\mathcal{N}(0, 1)$  as  $n \rightarrow \infty$ . Note that when  $n$  approaches a couple dozens, the distribution of  $\mathcal{Z}_n$  is very nearly  $\mathcal{N}(0, 1)$ . Given that  $\omega \in (0, 1)$ , let  $c(\omega)$  be the  $\omega$ -critical value for a standard normal variable  $\mathcal{Z}$ , that is,  $P(\mathcal{Z} \leq c(\omega)) = \omega$ . Then we have  $\lim_{n \rightarrow \infty} P(\mathcal{Z}_n \leq c(\omega)) = \omega$ . Hence, for a relatively large  $n$ ,

$$P\left(\frac{\sum_{i=1}^n \mathcal{Y}^i - n\mu_F}{\sigma_F \sqrt{n}} \leq c(\omega)\right) \approx \omega,$$

and hence,  $P(\sum_{i=1}^n \mathcal{Y}^i \leq \tilde{c}(\omega)) \approx \omega$ , where  $\tilde{c}(\omega) = c(\omega)\sigma_F \sqrt{n} + n\mu_F$ . In view of this result and Proposition 3.1, one can see that the set  $\mathcal{S}_{\mu,v}$  given in (11) with such a  $\tilde{c}(\omega)$  is an  $\omega$ -confidence uncertainty set of  $(\mu, V)$  when  $n$  is relatively large (say a couple dozen). We next consider the case where  $n$  is relatively small. Recall that  $\{\mathcal{Y}^i\}_{i=1}^n$  are i.i.d. and have  $F$ -distribution. Thus, we can apply simulation techniques (e.g. see [25]) to find a  $h(\omega)$  such that  $P(\sum_{i=1}^n \mathcal{Y}^i \leq h(\omega)) \approx \omega$ . In view of this relation and Proposition 3.1, we immediately see that the set  $\mathcal{S}_{\mu,v}$  given in (11) with  $\tilde{c}(\omega) = h(\omega)$  is an  $\omega$ -confidence uncertainty set of  $(\mu, V)$ . Therefore, for every  $\omega \in (0, 1)$ , one can apply the above statistical procedure to build an uncertainty set of  $(\mu, V)$  in the form of (11) with a  $\omega$ -confidence level.

From now on, we assume that for every  $\omega \in (0, 1)$ ,  $\mathcal{S}_{\mu,v}(\omega)$  given in (11), simply denoted by  $\mathcal{S}_{\mu,v}$ , is a ‘joint’ ellipsoidal uncertainty set of  $(\mu, V)$  with a  $\omega$ -confidence level. In view of (9), one has  $\sum_{i=1}^n \mathcal{Y}^i \geq 0$ . Moreover, we know from Theorem 2.1 that  $\sum_{i=1}^n \mathcal{Y}^i$  is a continuous random variable. Using these facts and Proposition 3.1, one can observe that the following property holds for  $\mathcal{S}_{\mu,v}$ :

$$\tilde{c}(\omega) > 0, \quad \text{if } \omega > 0. \tag{13}$$

We next consider several robust portfolio selection problems for the ‘joint’ ellipsoidal uncertainty set  $\mathcal{S}_{\mu,v}$ . Indeed, an investor’s position in the market can be described by a portfolio  $\phi \in \mathfrak{R}^n$ , where the  $i$ th component  $\phi_i$  represents the fraction of total wealth invested in the  $i$ th asset. The return  $r_\phi$  on the portfolio  $\phi$  is given by

$$r_\phi = r^T \phi = \mu^T \phi + f^T V \phi + \epsilon^T \phi \sim \mathcal{N}(\phi^T \mu, \phi^T (V^T F V + D) \phi), \tag{14}$$

and hence, the mean and variance of  $r_\phi$  are

$$E[r_\phi] = \phi^T \mu \quad \text{and} \quad \text{Var}[r_\phi] = \phi^T (V^T F V + D) \phi, \tag{15}$$

respectively. Generally, for any investment, there are costs associated with short-sale restrictions. On the other hand, as shown in Jagannathan and Ma [19], no short-sale restrictions can consistently reduce estimation errors on the covariance matrix. Therefore, we assume that short-sale restrictions are not imposed, that is,  $\phi \geq 0$ . Let

$$\Phi = \{\phi : e^T \phi = 1, \phi \geq 0\}. \tag{16}$$

The objective of the *robust maximum return* problem is to maximize the worst-case expected return subject to a constraint on the worst-case variance, that is, to solve the following problem:

$$\begin{aligned} \max_{\phi} \quad & \min_{(\mu, V) \in \mathcal{S}_{\mu,v}} E[r_\phi] \\ \text{s.t.} \quad & \max_{(\mu, V) \in \mathcal{S}_{\mu,v}} \text{Var}[r_\phi] \leq \lambda, \\ & \phi \in \Phi. \end{aligned} \tag{17}$$

A closely related problem, the *robust minimum variance* problem, is the ‘dual’ of (17). The objective of this problem is to minimize the worst-case variance of the portfolio subject to a constraint on the worst-case expected return on the portfolio. It can be formulated as

$$\begin{aligned} \min_{\phi} \quad & \max_{(\mu, V) \in \mathcal{S}_{\mu,v}} \text{Var}[r_\phi] \\ \text{s.t.} \quad & \min_{(\mu, V) \in \mathcal{S}_{\mu,v}} E[r_\phi] \geq \beta, \\ & \phi \in \Phi. \end{aligned} \tag{18}$$

We now address a drawback associated with problem (17). Let

$$\mathcal{S}_\mu = \{\mu : (\mu, V) \in \mathcal{S}_{\mu,v} \text{ for some } V\}, \quad \mathcal{S}_V = \{V : (\mu, V) \in \mathcal{S}_{\mu,v} \text{ for some } \mu\}.$$

In view of (15), we observe that (17) is equivalent to

$$\begin{aligned} \max_{\phi} \quad & \min_{(\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V} E[r_\phi] \\ \text{s.t.} \quad & \max_{(\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V} \text{Var}[r_\phi] \leq \lambda, \\ & \phi \in \Phi. \end{aligned} \tag{19}$$

Hence, the uncertainty set of  $(\mu, V)$  used in problem (17) is essentially  $\mathcal{S}_\mu \times \mathcal{S}_V$ . Recall that  $\mathcal{S}_{\mu,v}$  has a  $\omega$ -confidence level, and hence, we have

$$P((\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V) \geq P((\mu, V) \in \mathcal{S}_{\mu,v}) = \omega.$$

Using this relation, we immediately conclude that  $\mathcal{S}_\mu \times \mathcal{S}_V$  has at least a  $\omega$ -confidence level, but its actual confidence level is unknown and could be much higher than the desired  $\omega$ . Hence, problem (17) can be too conservative even though  $\mathcal{S}_{\mu,v}$  has the desired confidence level  $\omega$ . Using a similar argument, we see that problem (18) also has this drawback.

To combat the drawback of problem (17), we establish the following two propositions.

PROPOSITION 3.2 *Let*

$$\lambda^l = \min_{\phi \in \Phi} \max_{V \in \mathcal{S}_V} \text{Var}[r_\phi],$$

and let  $\phi_\lambda^*$  denote an optimal solution of problem (17) for any  $\lambda > \lambda^l$ . Then,  $\phi_\lambda^*$  is also an optimal solution of the following problem

$$\max_{\phi \in \Phi} \min_{(\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V} E[r_\phi] - \theta \text{Var}[r_\phi], \quad (20)$$

for some  $\theta \geq 0$ .

*Proof* Recall that problem (17) is equivalent to problem (19). It implies that  $\phi_\lambda^*$  is also an optimal solution of (19). Let

$$f(\phi) = \min_{(\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V} E[r_\phi], \quad g(\phi) = \max_{(\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V} \text{Var}[r_\phi].$$

In view of (15), we see that  $f(\phi)$  is concave and  $g(\phi)$  is convex over the convex compact set  $\Phi$ . For any  $\lambda > \lambda^l$ , we can observe that: (i) problem (19) is feasible and its optimal value is finite and (ii) there exists  $\phi^0 \in \Phi$  such that  $g(\phi^0) < \lambda$ . Hence, from Bertsekas [5, Proposition 5.3.1], we know that there exists at least one Lagrange multiplier  $\theta \geq 0$  such that  $\phi_\lambda^*$  solves problem (20) for such  $\theta$ , and the conclusion follows. ■

We observe that the converse of Proposition 3.2 also holds.

PROPOSITION 3.3 *Let  $\phi_\theta^*$  denote an optimal solution of problem (20) for any  $\theta \geq 0$ . Then,  $\phi_\theta^*$  is also an optimal solution of problem (17) for*

$$\lambda = \max_{V \in \mathcal{S}_V} \text{Var}[r_{\phi_\theta^*}]. \quad (21)$$

*Proof* We first observe that  $\phi_\theta^*$  is a feasible solution of problem (17) with  $\lambda$  given by (21). Now assume for a contradiction that there exists  $\phi \in \Phi$  such that

$$\max_{(\mu, V) \in \mathcal{S}_{\mu,v}} \text{Var}[r_\phi] \leq \lambda, \quad \min_{(\mu, V) \in \mathcal{S}_{\mu,v}} E[r_\phi] > \min_{(\mu, V) \in \mathcal{S}_{\mu,v}} E[r_{\phi_\theta^*}],$$

or equivalently,

$$\max_{V \in \mathcal{S}_V} \text{Var}[r_\phi] \leq \lambda, \quad \min_{\mu \in \mathcal{S}_\mu} E[r_\phi] > \min_{\mu \in \mathcal{S}_\mu} E[r_{\phi_\theta^*}]$$

due to (15). These relations together with (15) and (21) imply that for such  $\phi$ ,

$$\begin{aligned} \min_{(\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V} E[r_\phi] - \theta \text{Var}[r_\phi] &= \min_{\mu \in \mathcal{S}_\mu} E[r_\phi] - \theta \max_{V \in \mathcal{S}_V} \text{Var}[r_\phi], \\ &> \min_{\mu \in \mathcal{S}_\mu} E[r_{\phi_\theta^*}] - \theta \max_{V \in \mathcal{S}_V} \text{Var}[r_{\phi_\theta^*}], \\ &= \min_{(\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V} E[r_{\phi_\theta^*}] - \theta \text{Var}[r_{\phi_\theta^*}], \end{aligned}$$

which is a contradiction of the fact that  $\phi_\theta^*$  is an optimal solution of problem (20). Thus, the conclusion holds. ■

In view of Propositions 3.2 and 3.3, we conclude that problems (17) and (20) are equivalent. We easily observe that the conservativeness of problem (20) (or equivalently (17)) can be alleviated if we replace  $\mathcal{S}_\mu \times \mathcal{S}_V$  by  $\mathcal{S}_{\mu, v}$  in (20). This leads to the RMRAR problem with the uncertainty set  $\mathcal{S}_{\mu, v}$ :

$$\max_{\phi \in \Phi} \min_{(\mu, V) \in \mathcal{S}_{\mu, v}} E[r_\phi] - \theta \text{Var}[r_\phi], \tag{22}$$

where  $\theta \geq 0$  represents the risk-aversion parameter.

In contrast to problems (17) and (18), the RMRAR problem (22) has a clear advantage that the confidence level of its underlying uncertainty set is controllable. In the next section, we show that problem (22) can be reformulated as a cone programming problem.

#### 4. Cone programming reformulation

In this section, we show that the RMRAR problem (22) can be reformulated as a cone programming problem.

In view of (14), the RMRAR problem (22) can be written as

$$\max_{\phi \in \Phi} \left\{ \min_{(\mu, V) \in \mathcal{S}_{\mu, v}} \{ \mu^T \phi - \theta \phi^T V^T F V \phi \} - \theta \phi^T D \phi \right\}. \tag{23}$$

By introducing auxiliary variables  $v$  and  $t$ , problem (23) can be reformulated as

$$\begin{aligned} \max_{\phi, v, t} \quad & v - \theta t \\ \text{s.t.} \quad & \min_{(\mu, V) \in \mathcal{S}_{\mu, v}} \{ \mu^T \phi - \theta \phi^T V^T F V \phi \} \geq v, \\ & \phi^T D \phi \leq t, \\ & \phi \in \Phi. \end{aligned} \tag{24}$$

We next aim to reformulate the inequality

$$\min_{(\mu, V) \in \mathcal{S}_{\mu, v}} \{ \mu^T \phi - \theta \phi^T V^T F V \phi \} \geq v \tag{25}$$

as linear matrix inequalities (LMIs). Before proceeding, we introduce two lemmas that will be used subsequently.

The following lemma is about the  $\mathcal{S}$ -procedure. For a discussion of the  $\mathcal{S}$ -procedure and its applications, see Boyd *et al.* [6].

LEMMA 4.1 Let  $F_i(x) = x^T A_i x + 2b_i^T x + c_i$ ,  $i = 0, \dots, p$  be quadratic functions of  $x \in \mathfrak{R}^n$ . Then  $F_0(x) \leq 0$  for all  $x$  such that  $F_i(x) \leq 0$ ,  $i = 1, \dots, p$ , if there exists  $\tau_i \geq 0$  such that

$$\sum_{i=1}^p \tau_i \begin{pmatrix} c_i & b_i^T \\ b_i & A_i \end{pmatrix} - \begin{pmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{pmatrix} \succeq 0.$$

Moreover, if  $p = 1$  then the converse holds if there exists  $x_0 \in \mathfrak{R}^n$  such that  $F_1(x_0) < 0$ .

In the next lemma, we state one simple property of the standard Kronecker product, denoted by  $\otimes$ . For its proof, see [18].

LEMMA 4.2 If  $H \succeq 0$  and  $K \succeq 0$ , then  $H \otimes K \succeq 0$ .

We are now ready to show that the inequality (25) can be reformulated as LMIs.

LEMMA 4.3 Let  $S_{\mu,v}$  be an  $\omega$ -confidence uncertainty set given in (11) for  $\omega \in (0, 1)$ . Then, the inequality (25) is equivalent to the following LMIs

$$\begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau \eta - 2v \end{pmatrix} \succeq 0, \quad (26)$$

$$\begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} \succeq 0, \quad \tau \geq 0,$$

where

$$R = \begin{pmatrix} \frac{A^T A}{s_1^2} & & & \\ & \ddots & & \\ & & \frac{A^T A}{s_n^2} & \\ & & & \frac{A^T A}{s_n^2} \end{pmatrix} \in \mathfrak{R}^{[(m+1)n] \times [(m+1)n]}, \quad \eta = \sum_{i=1}^n \bar{x}_i^T \left( \frac{A^T A}{s_i^2} \right) \bar{x}_i - \tilde{c}(\omega), \quad (27)$$

$$h = \begin{pmatrix} -\frac{A^T A \bar{x}_1}{s_1^2} \\ \vdots \\ -\frac{A^T A \bar{x}_n}{s_n^2} \end{pmatrix} \in \mathfrak{R}^{(m+1)n}, \quad q = \begin{pmatrix} \phi_1 \\ 0 \\ \vdots \\ \phi_n \\ 0 \end{pmatrix} \in \mathfrak{R}^{(m+1)n} \quad (28)$$

(here,  $0$  denotes the  $m$ -dimensional zero vector).

*Proof* Given any  $(v, \theta, \phi) \in \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}^n$ , we define

$$H(\mu, V) = -\mu^T \phi + \theta \phi^T V^T F V \phi + v.$$

As in (5), let  $x_i = (\mu_i, V_{1i}, V_{2i}, \dots, V_{mi})^T$  for  $i = 1, \dots, n$ . Viewing  $H(\mu, V)$  as a function of  $x = (x_1, \dots, x_n) \in \mathfrak{R}^{(m+1)n}$ , we have

$$\frac{\partial H}{\partial x_i} = \begin{pmatrix} -\phi_i \\ 2\theta \phi_i F V \phi \end{pmatrix}, \quad \frac{\partial^2 H}{\partial x_i \partial x_j} = \begin{pmatrix} 0 & 0 \\ 0 & 2\theta \phi_i \phi_j F \end{pmatrix}, \quad i, j = 1, \dots, n.$$

Using these relations and performing the Taylor series expansion for  $H(\mu, V)$  at  $x=0$ , we obtain that

$$H(\mu, V) = \frac{1}{2} \sum_{i,j=1}^n x_i^T \begin{pmatrix} 0 & 0 \\ 0 & 2\theta\phi_i\phi_jF \end{pmatrix} x_j + \sum_{i=1}^n \begin{pmatrix} -\phi_i \\ 0 \end{pmatrix}^T x_i + v. \tag{29}$$

Note that  $\mathcal{S}_{\mu,v}$  is given by (11). It can be written as

$$\mathcal{S}_{\mu,v} = \left\{ (\mu, V) : \sum_{i=1}^n x_i^T \left( \frac{A^T A}{s_i^2} \right) x_i - 2 \sum_{i=1}^n \left( \frac{A^T A \bar{x}_i}{s_i^2} \right)^T x_i + \sum_{i=1}^n \bar{x}_i^T \left( \frac{A^T A}{s_i^2} \right) \bar{x}_i - \tilde{c}(\omega) \leq 0 \right\}. \tag{30}$$

In view of (13) and the fact that  $\omega > 0$ , we have  $\tilde{c}(\omega) > 0$ . Hence, we see that  $x = \bar{x}$  strictly satisfies the inequality given in (30). Using (29), (30) and Lemma 4.1, we conclude that  $H(\mu, V) \leq 0$  for all  $(\mu, V) \in \mathcal{S}_{\mu,v}$  if and only if there exists  $\tau \in \mathfrak{R}$  such that

$$\tau \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} - \begin{pmatrix} E & -q \\ -q^T & 2v \end{pmatrix} \succeq 0, \quad \tau \geq 0, \tag{31}$$

where  $R, \eta, h$  and  $q$  are defined in (27) and (28), respectively, and  $E$  is given by

$$E = (E_{ij}) \in \mathfrak{R}^{[(m+1)n] \times [(m+1)n]}, \quad E_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 2\theta\phi_i\phi_jF \end{pmatrix} \in \mathfrak{R}^{(m+1) \times (m+1)}, \quad i, j = 1, \dots, n.$$

In terms of the Kronecker product  $\otimes$ , we can rewrite  $E$  as

$$E = 2\theta(\phi\phi^T) \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}.$$

This identity together with Lemma 4.2 and the fact that  $F \succeq 0$  implies that (31) holds if and only if

$$\begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau \eta - 2v \end{pmatrix} \succeq 0, \quad S \succeq \phi\phi^T, \quad \tau \geq 0. \tag{32}$$

Using the Schur Complement Lemma, we further observe that (32) holds if and only if (26) holds. Thus, we see that  $H(\mu, V) \leq 0$  for all  $(\mu, V) \in \mathcal{S}_{\mu,v}$  if and only if (26) holds. Then the conclusion immediately follows from this result and the fact that the inequality (25) holds if and only if  $H(\mu, V) \leq 0$  for all  $(\mu, V) \in \mathcal{S}_{\mu,v}$ . ■

In the following theorem, we show that the RMRAR problem (22) can be reformulated as a cone programming problem.

**THEOREM 4.4** *Let  $\mathcal{S}_{\mu,v}$  be an  $\omega$ -confidence uncertainty set given in (11) for  $\omega \in (0, 1)$ . Then, the RMRAR problem (22) is equivalent to*

$$\begin{aligned} & \max_{\phi, S, \tau, v, t} \quad v - \theta t \\ \text{s.t.} \quad & \begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau \eta - 2v \end{pmatrix} \succeq 0, \\ & \begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} \succeq 0, \\ & \begin{pmatrix} 1 + t \\ 1 - t \\ 2D^{1/2}\phi \end{pmatrix} \in \mathcal{L}^{n+2}, \\ & \tau \geq 0, \quad \phi \in \Phi, \end{aligned} \tag{33}$$

where  $\Phi$ ,  $R$ ,  $\eta$ ,  $h$  and  $q$  are defined in (16), (27) and (28), respectively, and  $\mathcal{L}^k$  denotes the  $k$ -dimensional second-order cone given by

$$\mathcal{L}^k = \left\{ z \in \mathfrak{R}^k : z_1 \geq \sqrt{\sum_{i=2}^k z_i^2} \right\}.$$

*Proof* We observe that the inequality  $\phi^T D \phi \leq t$  is equivalent to the third constraint of (33), which together with Lemma 4.3 implies that (24) is equivalent to (33). The conclusion immediately follows from this result and the fact that (22) is equivalent to (24). ■

The following theorem establishes the solvability of problem (33).

**THEOREM 4.5** *Assume that  $0 \neq F \succeq 0$ ,  $\omega \in (0, 1)$  and  $\theta > 0$ . Then, problem (33) and its dual problem are both strictly feasible, and hence, both problems are solvable and the duality gap is zero.*

*Proof* Let  $\text{ri}(\cdot)$  denote the relative interior of the associated set. We first show that problem (33) is strictly feasible. In view of (16), we immediately see that  $\text{ri}(\Phi) \neq \emptyset$ . Let  $\phi^0 \in \text{ri}(\Phi)$ , and let  $t^0 \in \mathfrak{R}$  such that  $t^0 > (\phi^0)^T D \phi^0$ . Then we can observe that

$$\begin{pmatrix} 1 + t^0 \\ 1 - t^0 \\ 2D^{1/2}\phi^0 \end{pmatrix} \in \text{ri}(\mathcal{L}^{n+2}).$$

Let  $S^0 \in \mathfrak{R}^{n \times n}$  be such that  $S^0 \succ \phi^0 (\phi^0)^T$ . Then by the Schur Complement Lemma, one has

$$\begin{pmatrix} 1 & (\phi^0)^T \\ \phi^0 & S^0 \end{pmatrix} \succ 0.$$

Using the assumption that  $A$  has full column rank, we observe from (27) that  $R \succ 0$ . Hence, there exists a sufficiently large  $\tau^0 > 0$  such that

$$M \equiv \tau^0 R - 2\theta S^0 \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \succ 0. \tag{34}$$

Now, let  $\nu^0$  be sufficiently small such that

$$\tau^0 \eta - 2\nu^0 - (\tau^0 h + q^0)^T M^{-1}(\tau^0 h + q^0) > 0,$$

where  $q^0 = (\phi_1^0, 0, \dots, \phi_n^0, 0)^T \in \mathfrak{R}^{(m+1)n}$  (here, 0 denotes the  $m$ -dimensional zero vector). This, together with (34) and the Schur Complement Lemma implies that

$$\begin{pmatrix} \tau^0 R - 2\theta S^0 \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau^0 h + q^0 \\ (\tau^0 h + q^0)^T & \tau^0 \eta - 2\nu^0 \end{pmatrix} > 0.$$

Thus, we see that  $(\phi^0, S^0, \tau^0, \nu^0, t^0)$  is a strictly feasible point of problem (33).

We next show that the dual of problem (33) is also strictly feasible. Let

$$X^1 = \begin{pmatrix} X_{11}^1 & X_{12}^1 \\ X_{21}^1 & X_{22}^1 \end{pmatrix}, \quad X^2 = \begin{pmatrix} X_{11}^2 & X_{12}^2 \\ X_{21}^2 & X_{22}^2 \end{pmatrix}, \quad x^3 = \begin{pmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{pmatrix} \tag{35}$$

be the dual variables corresponding to the first three constraints of problem (33), respectively, where  $X_{11}^1 \in \mathfrak{R}^{[(m+1)n] \times [(m+1)n]}$ ,  $X_{12}^1 \in \mathfrak{R}^{(m+1)n}$ ,  $X_{22}^2 \in \mathfrak{R}^{n \times n}$ ,  $X_{21}^2, x_3^3 \in \mathfrak{R}^n$ ,  $X_{11}^2, X_{12}^2, x_1^3, x_2^3 \in \mathfrak{R}$ . Also, let  $x^4 \in \mathfrak{R}$  be the dual variable corresponding to the constraint  $e^T \phi = 1$ . Then, we see that the dual of problem (33) is

$$\begin{aligned} \min_{X^1, X^2, x^3, x^4} \quad & X_{11}^2 + x_1^3 + x_2^3 + x^4 \\ \text{s.t.} \quad & -2\Psi(X_{12}^1) - 2X_{21}^2 - 2D^{1/2}x_3^3 + x^4 e \geq 0, \\ & 2\theta \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X_{11}^1 - X_{22}^2 = 0, \\ & - \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot X^1 \geq 0, \\ & -x_1^3 + x_2^3 = -\theta, \\ & 2X_{22}^1 = 1, \\ & X^1 \geq 0, \quad X^2 \geq 0, \quad x^3 \in \mathcal{L}^{n+2}, \end{aligned} \tag{36}$$

where  $\Psi: \mathfrak{R}^{(m+1)n} \rightarrow \mathfrak{R}^n$  is defined as  $\Psi(x) = (x_1, x_{m+2}, \dots, x_{(n-2)(m+1)+1}, x_{(n-1)(m+1)+1})^T$  for every  $x \in \mathfrak{R}^{(m+1)n}$ , and

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X \equiv \left( \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \cdot X_{ij} \right) \in \mathfrak{R}^{n \times n} \tag{37}$$

for any  $X = (X_{ij}) \in \mathfrak{R}^{[(m+1)n] \times [(m+1)n]}$  with  $X_{ij} \in \mathfrak{R}^{(m+1) \times (m+1)}$  for  $i, j = 1, \dots, n$ . We now construct a strictly feasible solution  $(X^1, X^2, x^3, x^4)$  of the dual problem (36). Let  $x^3 = (\theta, 0, \dots, 0) \in \mathfrak{R}^{n+2}$ . It clearly satisfies the constraint  $-x_1^3 + x_2^3 = -\theta$ , and moreover,  $x^3 \in \text{ri}(\mathcal{L}^{n+2})$  due to  $\theta > 0$ .

Next, let

$$X^1 = \frac{1}{2(1+\gamma)} \left[ \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \\ 1 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \\ 1 \end{pmatrix}^T + \gamma I \right]. \quad (38)$$

In view of this identity and (35), one has  $X_{22}^1 = \frac{1}{2}$ . Since  $\omega > 0$ , we know from (13) that  $\tilde{c}(\omega) > 0$ . This, together with (27), (28) and (38) implies that

$$\begin{aligned} - \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot X^1 &= -\frac{1}{2(1+\gamma)} \left[ R \cdot \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T + 2h^T \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} + \eta + \gamma \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot I \right] \\ &= -\frac{1}{2(1+\gamma)} \left[ -\tilde{c}(\omega) + \gamma \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot I \right] > 0 \end{aligned}$$

and  $X^1 \succ 0$  for sufficiently small positive  $\gamma$ . Now, let

$$X_{22}^2 = 2\theta \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X_{11}^1. \quad (39)$$

We next show that  $X_{22}^2 \succ 0$ . Indeed, using (37) and the assumption that  $0 \neq F \geq 0$ , we obtain

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot I \succ 0. \quad (40)$$

Further, we have for every  $u \in \mathfrak{R}^n$ ,

$$\begin{aligned} u^T \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot \left[ \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T \right] u &= \sum_{i,j} \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \cdot (u_i u_j \bar{x}_i \bar{x}_j^T), \\ &= \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \cdot \left( \sum_{i,j} u_i u_j \bar{x}_i \bar{x}_j^T \right), \\ &= \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \cdot \left( \left( \sum_i u_i \bar{x}_i \right) \left( \sum_i u_i \bar{x}_i \right)^T \right) \geq 0, \end{aligned}$$

and hence,

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot \left[ \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T \right] \geq 0.$$

This, together with (38)–(40) and the assumption that  $\theta > 0$ , implies that  $X_{22}^2 \succ 0$ . Letting  $X_{12}^2 = 0$  and  $X_{11}^2 = 1$ , we immediately see that  $X^2 \succ 0$ . We also observe that for sufficiently large  $x^4$ ,  $(X^1, X^2, x^3, x^4)$  also strictly satisfies the first constraint of (36). Hence, it is a strictly feasible solution of the dual problem (36). The remaining proof directly follows from strong duality. ■

In view of Theorem 4.5, we conclude that problem (33) can be suitably solved by primal–dual interior point solvers (e.g. [27,29]).

## 5. Concluding remarks

In this article, we considered the factor model of the random asset returns. By exploring the correlations of the mean return vector  $\mu$  and factor loading matrix  $V$ , we proposed a statistical approach for constructing a ‘joint’ ellipsoidal uncertainty set  $\mathcal{S}_{\mu,v}$  for  $(\mu, V)$ . We further showed that the RMRAR problem with such an uncertainty set can be reformulated and solved as a cone programming problem. The computational results reported in [21,22] demonstrate that the robust portfolio determined by the RMRAR model with our ‘joint’ uncertainty set outperforms that with Goldfarb and Iyengar’s ‘separable’ uncertainty set [16] in terms of wealth growth rate and transaction cost; and moreover, our robust portfolio is fairly diversified, but Goldfarb and Iyengar’s is surprisingly highly non-diversified. It would be interesting to extend the results of this article to other robust portfolio selection models, for example, robust maximum Sharpe ratio and robust VaR models (see [16]).

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## References

- [1] A. Ben-Tal and A. Nemirovski, *Robust convex optimization*, Math. Oper. Res. 23 (1998), pp. 769–805.
- [2] A. Ben-Tal and A. Nemirovski, *Robust solutions of uncertain linear programs*, Oper. Res. Lett. 25 (1999), pp. 1–13.
- [3] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, Engineering Applications*, MPS-SIAM Series on Optimization, SIAM, Philadelphia, PA, 2001.
- [4] A. Ben-Tal, T. Margalit, and A. Nemirovski, *Robust modeling of multi-stage portfolio problems*, in *High Performance Optimization*, H. Frenk, K. Roos, T. Terlaky, and S. Zhang, eds., Kluwer Academic Press, Dordrecht, 2000, pp. 303–328.
- [5] D. Bertsekas, *Nonlinear Programming*, 2nd ed., Athena Scientific, New York, 1999.
- [6] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Studies in Applied Mathematics Vol. 15, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1994.
- [7] M. Broadie, *Computing efficient frontiers using estimated parameters*, Ann. Oper. Res. 45 (1993), pp. 21–58.
- [8] V.K. Chopra, *Improving optimization*, J. Invest. 8 (1993), pp. 51–59.
- [9] V.K. Chopra and W.T. Ziemba, *The effect of errors in means, variance and covariance on optimal portfolio choice*, J. Portfolio Manage. 19 (1993), pp. 6–11.
- [10] V. DeMiguel and F.J. Nogales, *Portfolio selection with robust estimation*, Oper. Res. 57 (2009), pp. 560–577.
- [11] V. DeMiguel, L. Garlappi, and R. Uppal, *Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy?*, Rev. Financ. Stud. 22 (2009), pp. 1915–1953.
- [12] L. El Ghaoui and H. Lebret, *Robust solutions to least-squares problems with uncertain data*, SIAM J. Matrix Anal. Appl. 18 (1997), pp. 1035–1064.
- [13] L. El Ghaoui, F. Outstry, and H. Lebret, *Robust solutions to uncertain semidefinite programs*, SIAM J. Optim. 9 (1998), pp. 33–52.
- [14] L. El Ghaoui, M. Oks, and F. Outstry, *Worst-case value-at-risk and robust portfolio optimization: a conic programming approach*, Oper. Res. 51 (2003), pp. 543–556.
- [15] E. Erdogan, D. Goldfarb, and G. Iyengar, *Robust portfolio management*, manuscript, Department of Industrial Engineering and Operations Research, Columbia University, New York, 2004.
- [16] D. Goldfarb and G. Iyengar, *Robust portfolio selection problems*, Math. Oper. Res. 28 (2003), pp. 1–38.
- [17] B.Y. Halldórsson and R.H. Tütüncü, *An interior-point method for a class of saddle problems*, J. Optim. Theory Appl. 116 (2004), pp. 559–590.
- [18] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.
- [19] R. Jagannathan and T. Ma, *Risk reduction in large portfolios: why imposing the wrong constraints helps*, J. Finance 58 (2003), pp. 1651–1684.
- [20] O. Kallenberg, *Foundations of Modern Probability*, Springer-Verlag, New York, 1997.
- [21] Z. Lu, *A new cone programming approach for robust portfolio selection*, Tech. Rep., Department of Mathematics, Simon Fraser University, Burnaby, BC, 2006.
- [22] Z. Lu, *A computational study on robust portfolio selection based on a joint ellipsoidal uncertainty set*, Math. Program., (2009), DOI: 10.1007/s10107-009-0271-Z.

- [23] H.M. Markowitz, *Portfolio selection*, J. Finance 7 (1952), pp. 77–91.
- [24] R.O. Michaud, *Efficient Asset Management: A practical Guide to Stock Portfolio Management and Asset Allocation*, Financial Management Association, Survey and Synthesis, HBS Press, Boston, MA, 1998.
- [25] S.M. Ross, *Simulation*, 3rd ed., Academic Press, London, 2006.
- [26] M.R. Spiegel, *Theory and Problems of Probability and Statistics*, McGraw-Hill, New York, 1992.
- [27] J.F. Sturm, *Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones*, Optim. Methods Softw. 11–12 (1999), pp. 625–653.
- [28] R.H. Tütüncü and M. Koenig, *Robust asset allocation*, Ann. Oper. Res. 132 (2004), pp. 157–187.
- [29] R.H. Tütüncü, K.C. Toh, and M.J. Todd, *Solving semidefinite-quadratic-linear programs using SDPT3*, Math. Program. 95 (2003), pp. 189–217.
- [30] S. Zhu and M. Fukushima, *Worst-case conditional value-at-risk with application to robust portfolio management*, Oper. Res. (in press).