A New Cone Programming Approach for Robust Portfolio Selection

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Abstract

The robust portfolio selection problems have recently been studied by several researchers (e.g., see [15, 14, 16, 24]). In their work, the "separable" uncertainty sets of the problem parameters (e.g., mean and covariance of the random returns) were considered. These uncertainty sets share two common properties: i) the actual confidence level of the uncertainty sets is unknown, and it can be much higher than the desired one; and ii) the uncertainty sets are fully or partially box-type. The associated consequences are that the resulting robust portfolios can be too conservative, and moreover, they are usually highly non-diversified as observed in the computational experiments conducted in this paper and [24]. To combat these drawbacks, we consider a factor model for the random asset returns. For this model, we introduce a "joint" ellipsoidal uncertainty set for the model parameters and show that it can be constructed as a confidence region associated with a statistical procedure applied to estimate the model parameters. We further show that the robust maximum risk-adjusted return (RMRAR) problem with this uncertainty set can be reformulated and solved as a cone programming problem. Computational experiments are also conducted to compare the performance of the robust portfolios determined by the RMRAR model with our "joint" uncertainty set and Goldfarb and Iyengar's "separable" uncertainty set proposed in the seminal paper [15]. The computational results demonstrate that our robust portfolio outperforms Goldfarb and Iyengar's in terms of wealth growth rate and transaction cost; and moreover, ours is fairly diversified, but Goldfarb and Iyengar's is surprisingly highly non-diversified.

Key words: Robust optimization, mean-variance portfolio selection, maximum risk-adjusted return portfolio selection, cone programming, linear regression.

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1 Introduction

Portfolio selection problem is concerned with determining a portfolio such that the "return" and "risk" of the portfolio have a favorable trade-off. The first mathematical model for portfolio selection problem was developed by Markowitz [19] five decades ago, in which an optimal or efficient portfolio can be identified by solving a convex quadratic program. In his model, the "return" and "risk" of a portfolio are measured by the mean and variance of the random portfolio return, respectively. Thus, the Markowitz portfolio model is also widely referred to as the mean-variance model.

Despite the theoretical elegance and importance of the mean-variance model, it continues to encounter skepticism among the investment practitioners. One of the main reasons is that the optimal portfolios determined by the mean-variance model are often sensitive to perturbations in the parameters of the problem (e.g., expected returns and the covariance matrix), and thus lead to large turnover ratios with periodic adjustments of the problem parameters; see for example Michaud [20]. Various aspects of this phenomenon have also been extensively studied in the literature, for example, see [9, 7, 8].

As a recently emerging modeling tool, robust optimization can incorporate the perturbations in the parameters of the problems into the decision making process. Generally speaking, robust optimization aims to find solutions to given optimization problems with uncertain problem parameters that will achieve good objective values for all or most of realizations of the uncertain problem parameters. For details, see [2, 3, 4, 11, 13]. Recently, robust optimization has been applied to model portfolio selection problems in order to alleviate the sensitivity of optimal portfolios to statistical errors in the estimates of problem parameters. For example, Goldfarb and Iyengar [15] considered a factor model for the random portfolio returns, and proposed some statistical procedures for constructing uncertainty sets for the model parameters. For these uncertainty sets, they showed that the robust portfolio selection problems can be reformulated as second-order cone programs. Subsequently, Erdoğan et al. [14] extended this method to robust index tracking and active portfolio management problems. Alternatively, Tütüncü and Koenig [24] (see also Halldórsson and Tütüncü [16]) considered a box-type uncertainty structure for the mean and covariance matrix of the assets returns. For this uncertainty structure, they showed that the robust portfolio selection problems can be formulated and solved as smooth saddle-point problems that involve semidefinite constraints. In addition, for finite uncertainty sets, Ben-Tal et al. [1] studied the robust formulations of multi-stage portfolio selection problems. Also, El Ghaoui et al. [12] considered the robust value-at-risk (VaR) problems given the partial information on the distribution of the returns, and they showed that these problems can be cast as semidefinite programs. Zhu and Fukushima [26] showed that the robust conditional value-at-risk (CVaR) problems can be reformulated as linear programs or second-order cone programs for some simple uncertainty structures of the distributions of the returns. Recently, DeMiguel and Nogales [10] proposed a novel approach for portfolio selection by minimizing certain robust estimators of portfolio risk. In their method, robust estimation and portfolio optimization are performed by solving a single nonlinear program.

The structure of uncertainty set is an important ingredient in formulating and solving robust portfolio selection problems. The "separable" uncertainty sets have been commonly used in the literatures. For example, Tütüncü and Koenig [24] (see also Halldórsson and Tütüncü [16]) proposed the box-type uncertainty sets $\mathcal{S}_m = \{\mu : \mu^L \leq \mu \leq \mu^U\}$ and $\mathcal{S}_v = \{\Sigma : \Sigma \succeq 0, \Sigma^L \leq \Sigma \leq \Sigma^U\}$ for the mean μ and covariance Σ of the asset return vector r, respectively. Here $A \succeq 0$ (resp. $\succ 0$) denotes that the matrix A is symmetric and positive semidefinite (resp. definite). In the seminal paper [15], Goldfarb and Iyengar studied a factor model for the asset return vector r in the form of

$$r = \mu + V^T f + \epsilon,$$

where μ is the mean return vector, f is the random factor return vector, V is the factor loading matrix and ϵ is the residual return vector (see Section 2 for details). They also proposed some uncertainty sets $\mathcal{S}_m$ and $\mathcal{S}_v$ for μ and V, respectively; in particular, $\mathcal{S}_m$ is a box (see (2)) and $\mathcal{S}_v$ is a Cartesian product of a bunch of ellipsoids (see (3)). It shall be mentioned that all "separable" uncertainty sets above share two common properties: i) viewed as a joint uncertainty set, the actual confidence level of $\mathcal{S}_m \times \mathcal{S}_v$ is unknown though $\mathcal{S}_m$ and/or $\mathcal{S}_v$ may have known confidence levels individually; and ii) $\mathcal{S}_m \times \mathcal{S}_v$ is fully or partially box-type. The associated consequences are that the resulting robust portfolio can be too conservative since the actual confidence level of $\mathcal{S}_m \times \mathcal{S}_v$ can be much higher than the desired one, or equivalently, the uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ can be too noisy; and moreover it is highly non-diversified as observed in the computational experiments conducted in this paper and [24]. These drawbacks will be further addressed in details in Sections 2 and 5.

In this paper, we also consider the same factor model for asset returns as studied in [15]. To combat the aforementioned drawbacks, we propose a "joint" ellipsoidal uncertainty set for the model parameters (μ, V), and show that it can be constructed as a confidence region associated with a statistical procedure applied to estimate (μ, V) for any desired confidence level. We further show that for this uncertainty set, the robust maximum risk-adjusted return (RMRAR) problem can be reformulated and solved as a cone programming problem. Computational experiments are also conducted to compare the performance of the robust portfolio determined by the RMRAR model with our "joint" uncertainty set and Goldfarb and Iyengar's "separable" uncertainty set proposed in the seminal paper [15]. The computational results demonstrate that our robust portfolio outperforms Goldfarb and Iyengar's in terms of wealth growth rate and transaction cost; and moreover, our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is surprisingly highly non-diversified.

The rest of this paper is organized as follows. In Section 2, we describe the factor model for asset returns that was studied in Goldfarb and Iyengar [15], and briefly review the statistical procedure proposed in [15] for constructing a "separable" uncertainty set of the model parameters. The associated drawbacks of this uncertainty set are also addressed. In Section 3, we introduce a "joint" uncertainty set for the model parameters, and propose a statistical procedure for constructing it for any desired confidence level. Several robust portfolio selection problems for this uncertainty set are also discussed. In Section 4, we show that for our "joint" uncertainty set, the RMRAR problem can be reformulated and solved as a cone programming problem. In Section 5, we report the computational results for the RMRAR models with our "joint" uncertainty set and Goldfarb and Iyengar's "separable" uncertainty set, respectively. Finally, we give some concluding remarks in Section 6.

2 Factor model and separable uncertainty sets

In this section, we first describe the factor model for asset returns that was studied in Goldfarb and Iyengar [15]. Then we briefly review the statistical procedure proposed in [15] for constructing a "separable" uncertainty set for the model parameters. The associated drawbacks of this uncertainty set are also addressed.

The following factor model for asset returns was studied in [15]. Suppose that a discrete-time market has n traded assets. The vector of asset returns over a single market period is denoted by $r \in \Re^n$. The returns on the assets in different market periods are assumed to be independent. The single period return r is assumed to be a random vector given by

$$r = \mu + V^T f + \epsilon, \tag{1}$$

where $\mu \in \Re^n$ is the vector of mean returns, $f \sim \mathcal{N}(0, F) \in \Re^m$ denotes the returns of the m factors driving the market, $V \in \Re^{m\times n}$ denotes the factor loading matrix of the n assets, and $\epsilon \sim \mathcal{N}(0, D) \in \Re^n$ is the vector of residual returns. Here $x \sim \mathcal{N}(\mu, \Sigma)$ denotes that x is a multivariate normal random variable with mean μ and covariance Σ. Further, it is assumed that D is a positive semidefinite diagonal matrix, and the residual return vector ϵ is independent of the factor return vector f. Thus, it can be seen from the above assumption that $r \sim \mathcal{N}(\mu, V^T F V + D)$.

Goldfarb and Iyengar [15] also studied a robust model for (1). In their model, the mean return vector μ is assumed to lie in the uncertainty set $\mathcal{S}_m$ given by

$$\mathcal{S}_m = \{\mu : \ \mu = \mu_0 + \xi, \ |\xi_i| \le \gamma_i, i = 1, \ldots, n\}, \tag{2}$$

and the factor loading matrix V is assumed to belong to the uncertainty set $\mathcal{S}_v$ given by

$$\mathcal{S}_v = \{V : \ V = V_0 + W, \ \|W_i\|_G \le \rho_i, i = 1, \ldots, n\}, \tag{3}$$

where W_i is the ith column of W and $\|w\|_G = \sqrt{w^T G w}$ denotes the elliptic norm of w with respect to a symmetric, positive definite matrix G.

Two analogous statistical procedures were proposed in [15] for constructing the above uncertainty sets $\mathcal{S}_m$ and $\mathcal{S}_v$. As observed in our computational experiments, the behavior of these uncertainty sets is almost identical. Therefore, we only briefly review their first statistical procedure as follows. Suppose the market data consists of asset returns $\{r^t : t = 1, \ldots, p\}$ and factor returns $\{f^t : t = 1, \ldots, p\}$ for p trading periods. Then the linear model (1) implies that

$$r_i^t = \mu_i + \sum_{j=1}^{m} V_{ji} f_j^t + \epsilon_i^t, \ i = 1, \ldots, n, \ t = 1, \ldots, p. \tag{4}$$

As in the typical linear regression analysis, it is assumed that $\{\epsilon_i^t : \; i = 1, \ldots, n, \; t = 1, \ldots, p\}$ are all independent normal random variables and $\epsilon_i^t \sim \mathcal{N}(0, \sigma_i^2)$ for all $t = 1, \ldots, p$. Now, let $B = (f^1, f^2, \ldots, f^p) \in \Re^{m \times p}$ denote the matrix of factor returns, and let $e \in \Re^p$ denote an all-one vector. Further, define

$$y_i = (r_i^1, r_i^2, \ldots, r_i^p)^T, \quad A = (e \; B^T), \quad x_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T, \quad \epsilon_i = (\epsilon_i^1, \ldots, \epsilon_i^p)^T \tag{5}$$

for $i = 1, \ldots, n$. Then (4) can be rewritten as

$$y_i = Ax_i + \epsilon_i, \; \forall i = 1, \ldots, n. \tag{6}$$

Due to $p \gg m$ usually in practice, it was assumed in [15] that the matrix A has full column rank. As a result, it follows from (6) that the least-squares estimate $\bar{x}_i$ of the true parameter x_i is given by

$$\bar{x}_i = (A^T A)^{-1} A^T y_i. \tag{7}$$

The following result has played a crucial role in constructing the uncertainty sets $\mathcal{S}_m$ and $\mathcal{S}_v$ in [15]. It will also be used to build a 'joint" ellipsoidal uncertainty set in Section 3.

Theorem 2.1 *Let s_i^2 be the unbiased estimate of σ_i^2 given by*

$$s_i^2 = \frac{\|y_i - A\bar{x}_i\|^2}{p - m - 1} \tag{8}$$

for $i = 1, \ldots, n$. Then the random variables

$$\mathscr{Y}^i = \frac{1}{(m+1)s_i^2} (\bar{x}_i - x_i)^T A^T A (\bar{x}_i - x_i), \; i = 1, \ldots, n, \tag{9}$$

are distributed according to the F-distribution with $m+1$ degrees of freedom in the numerator and $p - m - 1$ degrees of freedom in the denominator. Moreover, $\{\mathscr{Y}^i\}_{i=1}^n$ are independent.

Proof. The first statement was shown in [15] (see page 16 of [15]). We now prove the second statement. In view of (6) and (7), we obtain that

$$\bar{x}_i - x_i = (A^T A)^{-1} A^T \epsilon_i, \quad y_i - A\bar{x}_i = [I - A(A^T A)^{-1} A^T]\epsilon_i.$$

Using these relations and the assumption that $\{\epsilon_i : \; i = 1, \ldots, n\}$ are independent, we conclude that $\{\mathscr{Y}^i\}_{i=1}^n$ are independent. ∎

Let $\mathscr{F}_J$ denote the cumulative distribution function of the F-distribution with J degrees of freedom in the numerator and $p-m-1$ degrees of freedom in the denominator. Given any $\tilde{\omega} \in (0, 1)$, let $c_J(\tilde{\omega})$ be its $\tilde{\omega}$-critical value, i.e., $\mathscr{F}_J(c_J(\tilde{\omega})) = \tilde{\omega}$. Also, we let

$$\begin{aligned} \mathcal{S}^i(\tilde{\omega}) &= \left\{ x_i : \; (\bar{x}_i - x_i)^T A^T A (\bar{x}_i - x_i) \leq (m+1) c_{m+1}(\tilde{\omega}) s_i^2 \right\}, \; i = 1, \ldots, n, \qquad (10) \\ \mathcal{S}(\tilde{\omega}) &= \mathcal{S}^1(\tilde{\omega}) \times \mathcal{S}^2(\tilde{\omega}) \times \cdots \times \mathcal{S}^n(\tilde{\omega}). \end{aligned}$$

Using Theorem 2.1, Goldfarb and Iyengar showed that $\mathcal{S}(\tilde{\omega})$ is an $\tilde{\omega}^n$-confidence set of (μ, V). Let $\mathcal{S}_m(\tilde{\omega})$ and $\mathcal{S}_v(\tilde{\omega})$ denote the projection of $\mathcal{S}(\tilde{\omega})$ along μ, V, respectively. Their explicit expressions can be found in Section 5 of [15], and they are in the form of (2)-(3). Goldfarb and Iyengar [15] set $\mathcal{S}_m := \mathcal{S}_m(\tilde{\omega})$ and $\mathcal{S}_v := \mathcal{S}_m(\tilde{\omega})$, and viewed them as the $\tilde{\omega}^n$-confidence sets of μ and V, respectively. However, we immediately observe that

$$\mathrm{P}\left(\mu \in \mathcal{S}_m(\tilde{\omega})\right) \geq \mathrm{P}\left((\mu, V) \in \mathcal{S}(\tilde{\omega})\right) = \tilde{\omega}^n,$$

and similarly, $\mathrm{P}\left(V \in \mathcal{S}_v(\tilde{\omega})\right) \geq \tilde{\omega}^n$. Hence, $\mathcal{S}_m$ and $\mathcal{S}_v$ have at least $\tilde{\omega}^n$-confidence levels, but their actual confidence levels are unknown and can be much higher than $\tilde{\omega}^n$. Further, in view of the relation $\mathcal{S}(\tilde{\omega}) \subseteq \mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$, one has

$$\mathrm{P}\left((\mu, V) \in \mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})\right) \geq \mathrm{P}\left((\mu, V) \in \mathcal{S}(\tilde{\omega})\right) = \tilde{\omega}^n.$$

Thus $\mathcal{S}_m(\tilde{\omega}) \times \mathcal{S}_v(\tilde{\omega})$, as a joint uncertainty set of (μ, V), has at least $\tilde{\omega}^n$-confidence level. However, its actual confidence level is unknown and can be much higher than the desired one that is $\tilde{\omega}^n$. One immediate consequence is that the robust portfolio selection models based on such uncertainty sets $\mathcal{S}_m(\tilde{\omega})$ and $\mathcal{S}_v(\tilde{\omega})$ can be too conservative. In addition, we observe from computational experiments that the resulting robust portfolios are highly non-diversified, in other words, they concentrate only on a few assets. One possible interpretation of this phenomenon is that $\mathcal{S}_m(\tilde{\omega})$ has a box-type structure. To combat these drawbacks, we introduce a "joint" ellipsoidal uncertainty set for (μ, V) in Section 3, and show that it can be constructed by a statistical approach.

Before ending this section, we shall remark that in Goldfarb and Iyengar's robust factor model, some uncertainty structures were also proposed for the parameters F and D that are the covariances of factor and residual returns. Nevertheless, they are often assumed to be fixed in practical computations, and can be estimated by some standard statistical approaches (see Section 7 of [15]). For the brevity of presentation, we assume that F and D are fixed throughout the rest of paper. But we shall mention that the results of this paper can be extended to the case where F and D have the same uncertainty structures as described in [15].

3 Joint uncertainty set and robust portfolio selection models

In this section, we consider the same factor model for asset returns as described in Section 2. In particular, we first introduce a "joint" ellipsoidal uncertainty set for the model parameters (μ, V). Then we propose a statistical procedure for constructing it. Finally, we discuss several robust portfolio selection models for this uncertainty set.

Throughout this section, we assume that all notations are given in Section 2, unless explicitly defined otherwise.

Recall from Section 2 that the "separable" uncertainty set $\mathcal{S}_m(\tilde{\omega})\times\mathcal{S}_v(\tilde{\omega})$ of (μ, V) proposed in [15] has several drawbacks. To overcome these drawbacks, we consider a "joint" ellipsoidal uncertainty set of (μ, V) with ω-confidence level in the form of

$$\mathcal{S}_{\mu,v}(\omega) = \left\{ (\tilde{\mu}, \tilde{V}) \in \Re^n \times \Re^{m\times n} : \sum_{i=1}^{n} \frac{(\tilde{x}_i - \bar{x}_i)^T (A^T A)(\tilde{x}_i - \bar{x}_i)}{s_i^2} \leq (m+1)\tilde{c}(\omega) \right\} \tag{11}$$

for some $\tilde{c}(\omega)$, where $\tilde{x}_i = (\tilde{\mu}_i, \tilde{V}_{1i}, \tilde{V}_{2i}, \ldots, \tilde{V}_{mi})^T$ for $i = 1, \ldots, n$. Using the definition of $\{\mathscr{Y}^i\}_{i=1}^n$ (see (9)), we can observe that the following property holds for $\mathcal{S}_{\mu,v}(\omega)$.

Proposition 3.1 *$\mathcal{S}_{\mu,v}(\omega)$ is an ω-confidence uncertainty set of (μ, V) for some $\tilde{c}(\omega)$ if and only if $\mathrm{P}(\sum_{i=1}^{n} \mathscr{Y}^i \leq \tilde{c}(\omega)) = \omega$, that is, $\tilde{c}(\omega)$ is the ω-critical value of $\sum_{i=1}^{n} \mathscr{Y}^i$.*

We are now ready to propose a statistical procedure for constructing the aforementioned "joint" ellipsoidal uncertainty set for the parameters (μ, V). First, we know from Theorem 2.1 that the random variables $\{\mathscr{Y}^i\}_{i=1}^n$ have the F-distribution with $m+1$ degrees of freedom in the numerator and $p-m-1$ degrees of freedom in the denominator. It follows from a standard statistical result (e.g., see [22]) that their mean and standard deviation are

$$\mu_F = \frac{p-m-1}{p-m-3}, \qquad \sigma_F = \sqrt{\frac{2(p-m-1)^2(p-2)}{(m+1)(p-m-3)^2(p-m-5)}}, \tag{12}$$

respectively, provided that $p > m+5$, which often holds in practice. In view of Theorem 2.1, we also know that $\{\mathscr{Y}^i\}_{i=1}^n$ are *i.i.d.*. Using this fact and the central limit theorem (e.g., see [18]), we conclude that the distribution of the random variable

$$\mathscr{Z}_n = \frac{\sum_{i=1}^{n} \mathscr{Y}^i - n\mu_F}{\sigma_F \sqrt{n}}$$

converges towards the standard normal distribution $\mathcal{N}(0,1)$ as $n \to \infty$. Notice that when n approaches a couple dozen, the distribution of $\mathscr{Z}_n$ is very nearly $\mathcal{N}(0,1)$. Given an $\omega \in (0,1)$, let $c(\omega)$ be the ω-critical value for a standard normal variable $\mathscr{Z}$, i.e., $\mathrm{P}(\mathscr{Z} \leq c(\omega)) = \omega$. Then we have $\lim_{n\to\infty} \mathrm{P}(\mathscr{Z}_n \leq c(\omega)) = \omega$. Hence, for a relatively large n,

$$\mathrm{P}\left(\frac{\sum_{i=1}^{n} \mathscr{Y}^i - n\mu_F}{\sigma_F\sqrt{n}} \leq c(\omega) \right) \approx \omega,$$

and hence, $\mathrm{P}(\sum_{i=1}^{n} \mathscr{Y}^i \leq \tilde{c}(\omega)) \approx \omega$, where $\tilde{c}(\omega) = c(\omega)\sigma_F\sqrt{n} + n\mu_F$. In view of this result and Proposition 3.1, one can see that the set $\mathcal{S}_{\mu,V}$ given in (11) with such a $\tilde{c}(\omega)$ is an ω-confidence uncertainty set of (μ, V) when n is relatively large (say a couple dozen). We next consider the case where n is relatively small. Recall that $\{\mathscr{Y}^i\}_{i=1}^n$ are *i.i.d.* and have F-distribution. Thus, we can apply simulation techniques (e.g., see [21]) to find a $h(\omega)$ such that $\mathrm{P}(\sum_{i=1}^{n} \mathscr{Y}^i \leq h(\omega)) \approx \omega$. In view of this relation and Proposition 3.1, we immediately see that the set $\mathcal{S}_{\mu,V}$ given in (11) with $\tilde{c}(\omega) = h(\omega)$ is an ω-confidence uncertainty set of (μ, V). Therefore, for every $\omega \in (0,1)$, one can apply the above statistical procedure to build an uncertainty set of (μ, V) in the form of (11) with ω-confidence level.

From now on, we assume that for every $\omega \in (0,1)$, $\mathcal{S}_{\mu,v}(\omega)$ given in (11), simply denoted by $\mathcal{S}_{\mu,v}$, is a "joint" ellipsoidal uncertainty set of (μ, V) with ω-confidence level. In view of (9), one has $\sum_{i=1}^n \mathscr{Y}^i \geq 0$. Moreover, we know from Theorem 2.1 that $\sum_{i=1}^n \mathscr{Y}^i$ is a continuous random variable. Using these facts and Proposition 3.1, one can observe that the following property holds for $\mathcal{S}_{\mu,v}$:

$$\tilde{c}(\omega) > 0, \quad \text{if } \omega > 0. \tag{13}$$

We next consider several robust portfolio selection problems for the "joint" ellipsoidal uncertainty set $\mathcal{S}_{\mu,v}$. Indeed, an investor's position in market can be described by a portfolio $\phi \in \Re^n$, where the ith component ϕ_i represents the fraction of total wealth invested in ith asset. The return r_ϕ on the portfolio ϕ is given by

$$r_\phi = r^T\phi = \mu^T\phi + f^T V\phi + \epsilon^T\phi \sim \mathcal{N}(\phi^T\mu, \phi^T(V^T F V + D)\phi), \tag{14}$$

and hence, the mean and variance of r_ϕ are

$$\mathrm{E}[r_\phi] = \phi^T\mu, \qquad \mathrm{Var}[r_\phi] = \phi^T(V^T F V + D)\phi, \tag{15}$$

respectively. For convenience, we assume that short sales are not allowed, i.e., $\phi \geq 0$. Let

$$\Phi = \{\phi : \ e^T\phi = 1, \phi \geq 0\}. \tag{16}$$

The objective of the *robust maximum return* problem is to maximize the worst case expected return subject to a constraint on the worst case variance, i.e., to solve the following problem

$$\begin{array}{lll} \max\limits_{\phi} & \min\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{E}[r_\phi] & \\ \text{s.t.} & \max\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{Var}[r_\phi] \leq \lambda, & \\ & \phi \in \Phi. & \end{array} \tag{17}$$

A closely related problem, the *robust minimum variance* problem, is the "dual" of (17). The objective of this problem is to minimize the worst case variance of the portfolio subject to a constraint on the worst case expected return on the portfolio. It can be formulated as

$$\begin{array}{lll} \min\limits_{\phi} & \max\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{Var}[r_\phi] & \\ \text{s.t.} & \min\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{E}[r_\phi] \geq \beta, & \\ & \phi \in \Phi. & \end{array} \tag{18}$$

We now address a drawback associated with problem (17). Let

$$\mathcal{S}_\mu = \{\mu : \ (\mu, V) \in \mathcal{S}_{\mu,v} \text{ for some } V\}, \quad \mathcal{S}_V = \{V : \ (\mu, V) \in \mathcal{S}_{\mu,v} \text{ for some } \mu\}.$$

In view of (15), we observe that (17) is equivalent to

$$\begin{array}{ll} \max\limits_{\phi} & \min\limits_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_\phi] \\ \text{s.t.} & \max\limits_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{Var}[r_\phi] \le \lambda, \\ & \phi \in \Phi. \end{array} \tag{19}$$

Hence, the uncertainty set of (μ, V) used in problem (17) is essentially $\mathcal{S}_\mu \times \mathcal{S}_V$. Recall that $\mathcal{S}_{\mu,v}$ has ω-confidence level, and hence, we have

$$\mathrm{P}\left((\mu, V) \in \mathcal{S}_\mu \times \mathcal{S}_V\right) \ge \mathrm{P}\left((\mu, V) \in \mathcal{S}_{\mu,v}\right) = \omega.$$

Using this relation, we immediately conclude that $\mathcal{S}_\mu \times \mathcal{S}_V$ has at least ω-confidence level, but its actual confidence level is unknown and can be much higher than the desired ω. Hence, problem (17) can be too conservative even though $\mathcal{S}_{\mu,v}$ has the desired confidence level ω. Using a similar argument, we see that problem (18) also has this drawback.

To combat the drawback of problem (17), we establish the following two propositions.

Proposition 3.2 *Let*

$$\lambda^l = \min_{\phi\in\Phi} \max_{V\in\mathcal{S}_V} \mathrm{Var}[r_\phi],$$

and let ϕ^*_λ denote an optimal solution of problem (17) for any $\lambda > \lambda^l$. Then, ϕ^*_λ is also an optimal solution of the following problem

$$\max_{\phi\in\Phi} \min_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_\phi] - \theta \mathrm{Var}[r_\phi], \tag{20}$$

for some $\theta \ge 0$.

Proof. Recall that problem (17) is equivalent to problem (19). It implies that ϕ^*_λ is also an optimal solution of (19). Let

$$f(\phi) = \min_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_\phi], \quad g(\phi) = \max_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{Var}[r_\phi].$$

In view of (15), we see that $f(\phi)$ is concave and $g(\phi)$ is convex over the convex compact set Φ. For any $\lambda > \lambda^l$, we can observe that: i) problem (19) is feasible and its optimal value is finite; and ii) there exists a $\phi^0 \in \Phi$ such that $g(\phi^0) < \lambda$. Hence, by Proposition 5.3.1 of Bertsekas [5], we know that there exists at least one Lagrange multiplier $\theta \ge 0$ such that ϕ^*_λ solves problem (20) for such θ, and the conclusion follows. ∎

We observe that the converse of Proposition 3.2 also holds.

Proposition 3.3 *Let ϕ^*_θ denote an optimal solution of problem (20) for any $\theta \geq 0$. Then, ϕ^*_θ is also an optimal solution of problem (17) for*

$$\lambda = \max_{V \in \mathcal{S}_V} \mathrm{Var}[r_{\phi^*_\theta}]. \tag{21}$$

Proof. We first observe that ϕ^*_θ is a feasible solution of problem (17) with λ given by (21). Now assume for a contradiction that there exits a $\phi \in \Phi$ such that

$$\max_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{Var}[r_\phi] \leq \lambda, \quad \min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{E}[r_\phi] > \min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{E}[r_{\phi^*_\theta}],$$

or equivalently,

$$\max_{V\in\mathcal{S}_V} \mathrm{Var}[r_\phi] \leq \lambda, \quad \min_{\mu\in\mathcal{S}_\mu} \mathrm{E}[r_\phi] > \min_{\mu\in\mathcal{S}_\mu} \mathrm{E}[r_{\phi^*_\theta}]$$

due to (15). These relations together with (15) and (21) imply that for such ϕ,

$$\begin{aligned}
\min_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_\phi] - \theta\mathrm{Var}[r_\phi] &= \min_{\mu\in\mathcal{S}_\mu} \mathrm{E}[r_\phi] - \theta \max_{V\in\mathcal{S}_V} \mathrm{Var}[r_\phi], \\
&> \min_{\mu\in\mathcal{S}_\mu} \mathrm{E}[r_{\phi^*_\theta}] - \theta \max_{V\in\mathcal{S}_V} \mathrm{Var}[r_{\phi^*_\theta}], \\
&= \min_{(\mu,V)\in\mathcal{S}_\mu\times\mathcal{S}_V} \mathrm{E}[r_{\phi^*_\theta}] - \theta\mathrm{Var}[r_{\phi^*_\theta}],
\end{aligned}$$

which is a contradiction to the fact that ϕ^*_θ is an optimal solution of problem (20). Thus, the conclusion holds. ∎

In view of Propositions 3.2 and 3.3, we conclude that problems (17) and (20) are equivalent. We easily observe that the conservativeness of problem (20) (or equivalently (17)) can be alleviated if we replace $\mathcal{S}_\mu \times \mathcal{S}_V$ by $\mathcal{S}_{\mu,v}$ in (20). This leads to the *robust maximum risk-adjusted return (RMRAR)* problem with the uncertainty set $\mathcal{S}_{\mu,v}$:

$$\max_{\phi\in\Phi} \min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \mathrm{E}[r_\phi] - \theta\mathrm{Var}[r_\phi], \tag{22}$$

where $\theta \geq 0$ represents the risk-aversion parameter.

In contrast with problems (17) and (18), the RMRAR problem (22) has a clear advantage that the confidence level of its underlying uncertainty set is controllable. In next section, we show that problem (22) can be reformulated as a cone programming problem.

4 Cone programming reformulation

In this section, we show that the RMRAR problem (22) can be reformulated as a cone programming problem.

In view of (14), the RMRAR problem (22) can be written as

$$\max_{\phi\in\Phi} \left\{ \min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \left\{\mu^T\phi - \theta\phi^T V^T F V \phi\right\} - \theta\phi^T D \phi \right\}. \tag{23}$$

By introducing auxiliary variables ν and t, problem (23) can be reformulated as

$$\begin{array}{ll} \max\limits_{\phi,\nu,t} & \nu - \theta t \\ \text{s.t.} & \min\limits_{(\mu,V)\in\mathcal{S}_{\mu,v}} \left\{\mu^T\phi - \theta\phi^T V^T F V\phi\right\} \geq \nu, \\ & \phi^T D\phi \leq t, \\ & \phi \in \Phi. \end{array} \tag{24}$$

We next aim to reformulate the inequality

$$\min_{(\mu,V)\in\mathcal{S}_{\mu,v}} \left\{\mu^T\phi - \theta\phi^T V^T F V\phi\right\} \geq \nu \tag{25}$$

as linear matrix inequalities (LMIs). Before proceeding, we introduce two lemmas that will be used subsequently.

The following lemma is about $\mathscr{S}$-procedure. For a discussion of $\mathscr{S}$-procedure and its applications, see Boyd et al. [6].

Lemma 4.1 *Let $F_i(x) = x^T A_i x + 2b_i x + c_i$, $i = 0,\ldots,p$ be quadratic functions of $x \in \Re^n$. Then $F_0(x) \leq 0$ for all x such that $F_i(x) \leq 0$, $i = 1,\ldots,p$, if there exists $\tau_i \geq 0$ such that*

$$\sum_{i=1}^{p} \tau_i \begin{pmatrix} c_i & b_i^T \\ b_i & A_i \end{pmatrix} - \begin{pmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{pmatrix} \succeq 0.$$

Moreover, if $p = 1$ then the converse holds if there exists $x_0 \in \Re^n$ such that $F_1(x_0) < 0$.

In the next lemma, we state one simple property of the standard Kronecker product, denoted by $\otimes$. For its proof, see [17].

Lemma 4.2 *If $H \succeq 0$ and $K \succeq 0$, then $H \otimes K \succeq 0$.*

We are now ready to show that the inequality (25) can be reformulated as LMIs.

Lemma 4.3 *Let $\mathcal{S}_{\mu,v}$ be an ω-confidence uncertainty set given in (11) for $\omega \in (0,1)$. Then, the inequality (25) is equivalent to the following LMIs*

$$\begin{array}{l} \begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau\eta - 2\nu \end{pmatrix} \succeq 0, \\ \begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} \succeq 0, \quad \tau \geq 0, \end{array} \tag{26}$$

where

$$R = \begin{pmatrix} \frac{A^TA}{s_1^2} & & \\ & \ddots & \\ & & \frac{A^TA}{s_n^2} \end{pmatrix} \in \Re^{[(m+1)n]\times[(m+1)n]}, \quad \eta = \sum_{i=1}^n \bar{x}_i^T \left(\frac{A^TA}{s_i^2}\right) \bar{x}_i - \tilde{c}(\omega), \tag{27}$$

$$h = \begin{pmatrix} -\frac{A^TA\bar{x}_1}{s_1^2} \\ \vdots \\ -\frac{A^TA\bar{x}_n}{s_n^2} \end{pmatrix} \in \Re^{(m+1)n}, \quad q = \begin{pmatrix} \phi_1 \\ 0 \\ \vdots \\ \phi_n \\ 0 \end{pmatrix} \in \Re^{(m+1)n} \tag{28}$$

(here, 0 *denotes the* m*-dimensional zero vector).*

Proof. Given any $(\nu, \theta, \phi) \in \Re \times \Re \times \Re^n$, we define

$$H(\mu, V) = -\mu^T\phi + \theta\phi^T V^T F V\phi + \nu.$$

As in (5), let $x_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T$ for $i = 1, \ldots, n$. Viewing $H(\mu, V)$ as a function of $x = (x_1, \ldots, x_n) \in \Re^{(m+1)n}$, we have

$$\frac{\partial H}{\partial x_i} = \begin{pmatrix} -\phi_i \\ 2\theta\phi_i F V\phi \end{pmatrix}, \quad \frac{\partial^2 H}{\partial x_i \partial x_j} = \begin{pmatrix} 0 & 0 \\ 0 & 2\theta\phi_i\phi_j F \end{pmatrix}, \; i, j = 1, \ldots, n.$$

Using these relations and performing the Taylor series expansion for $H(\mu, V)$ at $x = 0$, we obtain that

$$H(\mu, V) = \frac{1}{2}\sum_{i,j=1}^n x_i^T \begin{pmatrix} 0 & 0 \\ 0 & 2\theta\phi_i\phi_j F \end{pmatrix} x_j + \sum_{i=1}^n \begin{pmatrix} -\phi_i \\ 0 \end{pmatrix}^T x_i + \nu. \tag{29}$$

Notice that $\mathcal{S}_{\mu,v}$ is given by (11). It can be written as

$$\mathcal{S}_{\mu,v} = \left\{(\mu, V): \; \sum_{i=1}^n x_i^T \left(\frac{A^TA}{s_i^2}\right) x_i - 2\sum_{i=1}^n \left(\frac{A^TA\bar{x}_i}{s_i^2}\right)^T x_i + \sum_{i=1}^n \bar{x}_i^T \left(\frac{A^TA}{s_i^2}\right) \bar{x}_i - \tilde{c}(\omega) \leq 0\right\}. \tag{30}$$

In view of (13) and the fact that $\omega > 0$, we have $\tilde{c}(\omega) > 0$. Hence, we see that $x = \bar{x}$ strictly satisfies the inequality given in (30). Using (29), (30) and Lemma 4.1, we conclude that $H(\mu, V) \leq 0$ for all $(\mu, V) \in \mathcal{S}_{\mu,v}$ if and only if there exists a $\tau \in \Re$ such that

$$\tau \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} - \begin{pmatrix} E & -q \\ -q^T & 2\nu \end{pmatrix} \succeq 0, \quad \tau \geq 0, \tag{31}$$

where R, η, h and q are defined in (27) and (28), respectively, and E is given by

$$E = (E_{ij}) \in \Re^{[(m+1)n]\times[(m+1)n]}, \quad E_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 2\theta\phi_i\phi_j F \end{pmatrix} \in \Re^{(m+1)\times(m+1)}, \; i, j = 1, \ldots, n.$$

In terms of Kronecker product $\otimes$, we can rewrite E as

$$E = 2\theta(\phi\phi^T) \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}.$$

This identity together with Lemma 4.2 and the fact that $F \succeq 0$, implies that (31) holds if and only if

$$\begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau\eta - 2\nu \end{pmatrix} \succeq 0, \quad S \succeq \phi\phi^T, \quad \tau \geq 0. \tag{32}$$

Using Schur Complement Lemma, we further observe that (32) holds if and only if (26) holds. Thus, we see that $H(\mu, V) \leq 0$ for all $(\mu, V) \in \mathcal{S}_{\mu,v}$ if and only if (26) holds. Then the conclusion immediately follows from this result and the fact that the inequality (25) holds if and only if $H(\mu, V) \leq 0$ for all $(\mu, V) \in \mathcal{S}_{\mu,v}$. ■

In the following theorem, we show that the RMRAR problem (22) can be reformulated as a cone programming problem.

Theorem 4.4 *Let $\mathcal{S}_{\mu,v}$ be an ω-confidence uncertainty set given in (11) for $\omega \in (0,1)$. Then, the RMRAR problem (22) is equivalent to*

$$\begin{array}{lrcl} \max\limits_{\phi,S,\tau,\nu,t} & \nu - \theta t & & \\ s.t. & \begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau\eta - 2\nu \end{pmatrix} & \succeq & 0, \\ & \begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} & \succeq & 0, \\ & \begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2}\phi \end{pmatrix} & \in & \mathcal{L}^{n+2}, \\ & \tau \geq 0, \quad \phi & \in & \Phi, \end{array} \tag{33}$$

where Φ, R, η, h and q are defined in (16), (27) and (28), respectively, and $\mathcal{L}^k$ denotes the k-dimensional second-order cone given by

$$\mathcal{L}^k = \left\{ z \in \Re^k : \; z_1 \geq \sqrt{\sum_{i=2}^{k} z_i^2} \right\}.$$

Proof. We observe that the inequality $\phi^T D\phi \leq t$ is equivalent to the third constraint of (33), which together with Lemma 4.3, implies that (24) is equivalent to (33). The conclusion immediately follows from this result and the fact that (22) is equivalent to (24). ■

The following theorem establishes the solvability of problem (33). Its proof is given in the appendix.

Theorem 4.5 *Assume that $0 \neq F \succeq 0$, $\omega \in (0,1)$ and $\theta > 0$. Then, problem (33) and its dual problem are both strictly feasible, and hence, both problems are solvable and the duality gap is zero.*

In view of Theorem 4.5, we conclude that problem (33) can be suitably solved by primal-dual interior point solvers (e.g., SeDuMi [23] and SDPT3 [25]).

5 Computational results

In this section, we present computational experiments on the RMRAR models. We conduct two types of computational tests. The first type of tests are based on simulated data, and the second type of tests use real market data. The main objective of these computational tests is to compare the performance of the RMRAR models with our "joint" uncertainty set (11) and Goldfarb and Iyengar's "separable" uncertainty set (2)-(3). All computations are performed using SeDuMi V1.1R2 [23]. Throughout this section, the symbols "LROB" and "GIROB" are used to label the robust portfolios determined by the RMRAR models with our "joint" and Goldfarb and Iyengar's "separable" uncertainty sets, respectively. The following terminology will also be used in this section.

Definition 1 *The diversification number of a portfolio is defined as the number of its components that are above* 1%.

5.1 Computational results for simulated data

In this subsection, we conduct computational tests for simulated data. The data is generated in the same manner as described in Section 7 of [15]. Indeed, we fix the number of assets $n = 50$ and the number of factors $m = 5$. A symmetric positive definite factor covariance matrix F is randomly generated, and it is assumed to be certain. The nominal factor loading matrix V is also randomly generated. The covariance matrix D of the residual returns ϵ is assumed to be certain and set to $D = 0.1 \operatorname{diag}(V^T F V)$, that is, the linear model explains 90% of the asset variance. The nominal asset returns $\mu \in \Re^n$ are chosen independently according to a uniform distribution on $[0.5\%, 1.5\%]$. Finally, we generate a sequence of asset and factor return vectors r and f according to the normal distributions $\mathcal{N}(\mu, V^T F V + D)$ and $\mathcal{N}(0, F)$ for an investment period of length $p = 90$, respectively. We also randomly generate the asset returns, denoted by $R \in \Re^{n \times p}$, for next period of length p according to $\mathcal{N}(\mu, V^T F V + D)$. In addition, given a desired confidence level $\omega > 0$, our "joint" uncertainty set $\mathcal{S}_{\mu,v}$ is built as in Section 3, and Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is built as in Section 2 with $\tilde{\omega} = \omega^{1/n}$. As discussed in Section 2, $\mathcal{S}_m \times \mathcal{S}_v$ has at least ω-confidence level, but its actual confidence level is unknown.

Let ϕ_r be a robust portfolio computed from the data of the current period. Suppose that ϕ_r is held constant for the investment over next period. The *wealth growth rate* of ϕ_r over next period is defined as

$$(\Pi_{1 \leq k \leq p}(e + R_k))^T \phi_r - 1, \tag{34}$$

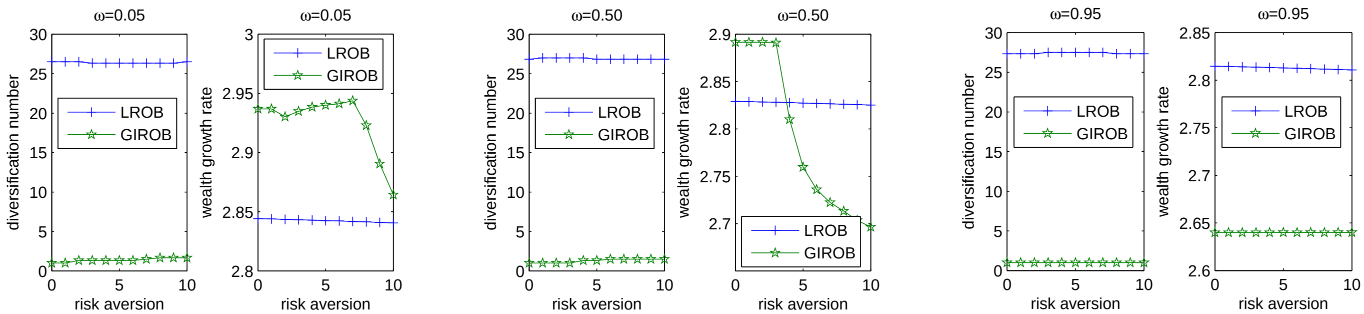

Figure 1: Performance of portfolios for $\omega = 0.05$, 0.50 and 0.95.

where $e \in \Re^n$ denotes the all-one vector and $R_k \in \Re^n$ denotes the kth column of R for $k = 1, \ldots, p$.

We next report the performance of the RMRAR models with our "joint" uncertainty set (11) and Goldfarb and Iyengar's "separable" uncertainty set described in (2)-(3) as the risk aversion parameter θ ranges from 0 to 10. The computational results averaged over 10 randomly generated instances are shown in Figure 1 that consists of three groups of plots for $\omega = 0.05, 0.50, 0.95$, respectively. In each of these three groups, the left plot is about the diversification number of the robust portfolios, and the right plot is about the wealth growth rate of the robust portfolios over next period. We first observe that our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified. Indeed, for $\omega = 0.05, 0.50, 0.95$, the diversification number of our robust portfolio is around 26, and that of Goldfarb and Iyengar's is around one or two. One possible interpretation of this phenomenon is that our uncertainty set $\mathcal{S}_{\mu,v}$ is ellipsoidal, but Goldfarb and Iyengar's uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is partially box-type. It seems that the ellipsoidal uncertainty structure tends to produce more diversified robust portfolio than does the fully or partially box-type one. In addition, we observe that for $\omega = 0.05$ or $\omega = 0.50$ with a relatively small θ, the wealth growth rate of our robust portfolio is lower than that of Goldfarb and Iyengar's. But for $\omega = 0.95$ or $\omega = 0.50$ with a relatively large θ, our wealth growth rate is higher than Goldfarb and Iyengar's. This phenomenon is actually not surprising. Indeed, we know that $\mathcal{S}_{\mu,v}$ has confidence ω while $\mathcal{S}_m \times \mathcal{S}_v$ has at least ω confidence, but its actual confidence level can be much higher than ω. Hence for a small ω, the associated model with $\mathcal{S}_m \times \mathcal{S}_v$ can be more robust than that with $\mathcal{S}_{\mu,v}$. However, for a relatively large ω, the uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ can be over confident, and its corresponding robust model can be conservative. This phenomenon becomes more prominent as the risk aversion parameter θ gets larger.

5.2 Computational results for real market data

In this subsection, we perform experiments on real market data for the RMRAR models with our "joint" and Goldfarb and Iyengar's "separable" uncertainty sets. The universe of assets that are chosen for investment are those ranked at the top of each of 10 industry categories by Fortune 500 in 2006. In total there are $n = 47$ assets in this set (see Table 1). The set of

Table 1: Assets

Aerospace and Defense		Telecommunications	
BA	Boeing Corp.	VZ	Verizon Communications
UTX	United Technologies	T	AT&T
LMT	Lockheed Martin	S	Sprint Nextel
NOC	Northrop Grumman	CMCSK	Comcast
HON	Honeywell Intl.	BLS	BellSouth
Semiconductors and Other Electronic Components		**Computer Software**	
INTC	Intel Corp.	MSFT	Microsoft
TXN	Texas Instruments	ORCL	Oracle
SANM	Sanmina-SCI	CA	CA
SLR	Solectron	ERTS	Electronic Arts
JBL	Jabil Circuit	SYMC	Symantec
Computers and Office Equipment		**Pharmaceuticals**	
IBM	Intl. Business Machines	PFE	Pfizer
HPQ	Hewlett-Packard	JNJ	Johnson & Johnson
DELL	Dell	ABT	Abbott Laboratories
XRX	Xerox	MRK	Merck
AAPL	Apple Computer	BMY	Bristol-Myers Squibb
Network and Other Communications Equipment		**Chemicals**	
MOT	Motorola	DOW	Dow Chemical
CSCO	Cisco Systems	DD	DuPon
LU	Lucent Technologies	LYO	Lyondell Chemical
QCOM	Qualcomm	PPG	PPG Industries
Electronics and Electrical Equipment		**Utilities (Gas & Electric)**	
EMR	Emerson Electric	DUK	Duke Energy
WHR	Whirlpool	D	Dominion Resources
ROK	Rockwell Automation	EXC	Exelon
SPW	SPX	SO	Southern
		PEG	Public Service Enterprise Group

factors are 10 major market indices (see Table 2). The data sequence consists of daily asset returns from July 25, 2002 through May 10, 2006. It shall be mentioned that the data used in this experiment was collected on May 11, 2006. The most recent data available at that time was the one on May 10, 2006.

A complete description of our experimental procedure is as follows. The entire data sequence is divided into investment periods of length $p = 90$ days. For each investment period t, the factor covariance matrix F is computed based on the factor returns of the previous p trading days, and the variance d_i of the residual return is set to $d_i = s_i^2$, where s_i^2 is given in Section 1. In addition, given a desired confidence level $\omega > 0$, our "joint" uncertainty set $\mathcal{S}_{\mu,v}$ is built as in Section 3, and Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is built as in Section 2 with $\tilde{\omega} = \omega^{1/n}$. The robust portfolios are then obtained by solving the RMRAR models with these uncertainty sets, and they are held constant for the investment at each period t.

Since a block of data of length $p = 90$ is required to construct uncertainty sets or estimate the parameters, the first investment period indexed by $t = 1$ starts from $(p+1)$th day. The time period July 25, 2002 – May 10, 2006 contains 11 periods of length $p = 90$, and hence in all there are 10 investment periods. Given a sequence of portfolios $\{\phi^t\}_{t=1}^{10}$, the corresponding *overall wealth growth rate* is defined as

$$\Pi_{1\le t\le 10}\left[\Pi_{tp\le k\le (t+1)p}(e + r_k)\right]^T \phi_r^t - 1,$$

Table 2: Factors

DJCMP65	Dow Jones Composite 65 Stock Average
DJINDUS	Dow Jones Industrials
DJUTILS	Dow Jones Utilities
DJTRSPT	Dow Jones Transportation
FRUSSL2	Russell 2000
NASA100	Nasdaq 100
NASCOMP	Nasdaq Composite
NYSEALL	NYSE Composite
S&PCOMP	S&P 500 Composite
WILEQTY	Dow Jones Wilshire 5000 Composite

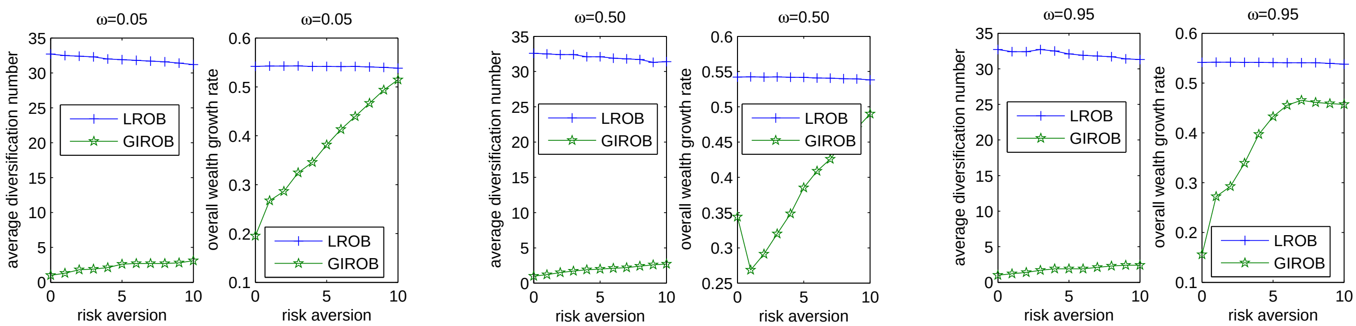

Figure 2: Performance of portfolios for $\omega = 0.05$, 0.50 and 0.95.

and the *average diversification number* is defined as $\sum_{t=1}^{10} I(\phi^t)/10$, where $I(\phi^t)$ denotes the diversification number of the portfolio ϕ^t.

We now report the performance of the RMRAR models with our "joint" uncertainty set $\mathcal{S}_{\mu,v}$, and Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ as the risk aversion parameter θ ranges from 0 to 10^4. The computational results for the confidence level $\omega = 0.05, 0.50, 0.95$ are shown in Figure 2 that consists of three group of plots. In each of these groups, the left plot is about the average diversification number of robust portfolios, and the second plot is about the overall wealth growth rate over next 10 periods of the investment using robust portfolios. We observe that our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified. Also, the overall wealth growth rate of the investment based on our robust portfolio is higher than that using Goldfarb and Iyengar's robust portfolio.

The realization cost is another natural concern for any investment strategy. We next compare the cost of implementing the above investment strategies. For a sequence of portfolios $\{\phi^t\}_{t=1}^{10}$, its *average transaction cost* is defined as $\sum_{t=2}^{10} \|\phi^t - \phi^{t-1}\|_1/9$ (see also the discussion in [15]). In Figure 3 we report the average transaction costs of the investments using the robust portfolios for the confidence levels $\omega = 0.05, 0.50, 0.95$, respectively. We observe that the investment based on our robust portfolio incurs lower average transaction cost than that using Goldfarb and Iyengar's robust portfolio.

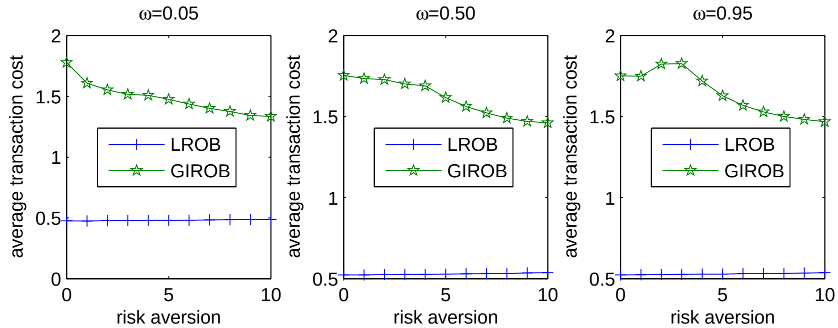

Figure 3: Average cost of portfolios for $\omega = 0.05$, 0.50 and 0.95.

6 Concluding Remarks

In this paper, we considered the factor model of the random asset returns. By exploring the correlations of the mean return vector μ and factor loading matrix V, we proposed a statistical approach for constructing a "joint" ellipsoidal uncertainty set $\mathcal{S}_{\mu,v}$ for (μ, V). We further showed that the robust maximum risk-adjusted return (RMRAR) problem with such a uncertainty set can be reformulated and solved as a cone programming problem. The computational results reported in this paper demonstrate that the robust portfolio determined by the RMRAR model with our "joint" uncertainty set outperforms that with Goldfarb and Iyengar's "separable" uncertainty set [15] in terms of wealth growth rate and transaction cost; and moreover, our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is surprisingly highly non-diversified. It would be interesting to extend the results of this paper to other robust portfolio selection models, e.g., robust maximum Sharpe ratio and robust value-at-risk models (see [15]).

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Appendix

In this section, we provide proof for Theorem 4.5.

Proof of Theorem 4.5: Let $\mathrm{ri}(\cdot)$ denote the relative interior of the associated set. We first show that problem (33) is strictly feasible. In view of (16), we immediately see that

$\mathrm{ri}(\Phi) \neq \emptyset$. Let $\phi^0 \in \mathrm{ri}(\Phi)$, and let $t^0 \in \Re$ such that $t^0 > (\phi^0)^T D\phi^0$. Then we can observe that

$$\begin{pmatrix} 1+t^0 \\ 1-t^0 \\ 2D^{1/2}\phi^0 \end{pmatrix} \in \mathrm{ri}(\mathcal{L}^{n+2})$$

Let $S^0 \in \Re^{n\times n}$ be such that $S^0 \succ \phi^0(\phi^0)^T$. Then by Schur Complement Lemma, one has

$$\begin{pmatrix} 1 & (\phi^0)^T \\ \phi^0 & S^0 \end{pmatrix} \succ 0.$$

Using the assumption that A has full column rank, we observe from (27) that $R \succ 0$. Hence, there exists a sufficiently large $\tau^0 > 0$ such that

$$M \equiv \tau^0 R - 2\theta S^0 \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \succ 0. \tag{35}$$

Now, let ν^0 be sufficiently small such that

$$\tau^0\eta - 2\nu^0 - (\tau^0 h + q^0)^T M^{-1}(\tau^0 h + q^0) > 0,$$

where $q^0 = (\phi_1^0, 0, \ldots, \phi_n^0, 0)^T \in \Re^{(m+1)n}$ (here, 0 denotes the m-dimensional zero vector). This together with (35) and Schur Complement Lemma, implies that

$$\begin{pmatrix} \tau^0 R - 2\theta S^0 \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau^0 h + q^0 \\ (\tau^0 h + q^0)^T & \tau^0\eta - 2\nu^0 \end{pmatrix} \succ 0.$$

Thus, we see that $(\phi^0, S^0, \tau^0, \nu^0, t^0)$ is a strictly feasible point of problem (33).

We next show that the dual of problem (33) is also strictly feasible. Let

$$X^1 = \begin{pmatrix} X^1_{11} & X^1_{12} \\ X^1_{21} & X^1_{22} \end{pmatrix}, \quad X^2 = \begin{pmatrix} X^2_{11} & X^2_{12} \\ X^2_{21} & X^2_{22} \end{pmatrix}, \quad x^3 = \begin{pmatrix} x^3_1 \\ x^3_2 \\ x^3_3 \end{pmatrix} \tag{36}$$

be the dual variables corresponding to the first three constraints of problem (33), respectively, where $X^1_{11} \in \Re^{[(m+1)n]\times[(m+1)n]}$, $X^1_{12} \in \Re^{(m+1)n}$, $X^2_{22} \in \Re^{n\times n}$, X^2_{21}, $x^3_3 \in \Re^n$, X^1_{22}, X^2_{11}, x^3_1, $x^3_2 \in \Re$. Also, let $x^4 \in \Re$ be the dual variable corresponding to the constraint $e^T\phi = 1$. Then, we see that the dual of problem (33) is

$$\begin{array}{lrcl} \min\limits_{X^1,X^2,x^3,x^4} & X^2_{11} + x^3_1 + x^3_2 + x^4 & & \\ \text{s.t.} & -2\Psi(X^1_{12}) - 2X^2_{21} - 2D^{1/2}x^3_3 + x^4 e & \geq & 0, \\ & 2\theta \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X^1_{11} - X^2_{22} & = & 0, \\ & -\begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \bullet X^1 & \geq & 0, \\ & -x^3_1 + x^3_2 & = & -\theta, \\ & 2X^1_{22} & = & 1, \\ & X^1 \succeq 0,\ X^2 \succeq 0,\ x^3 & \in & \mathcal{L}^{n+2}, \end{array} \tag{37}$$

where $\Psi : \Re^{(m+1)n} \rightarrow \Re^n$ is defined as $\Psi(x) = (x_1, x_{m+2}, \ldots, x_{(n-2)(m+1)+1}, x_{(n-1)(m+1)+1})^T$ for every $x \in \Re^{(m+1)n}$, and

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X \equiv \left(\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bullet X_{ij} \right) \in \Re^{n\times n} \tag{38}$$

for any $X = (X_{ij}) \in \Re^{[(m+1)n]\times[(m+1)n]}$ with $X_{ij} \in \Re^{(m+1)\times(m+1)}$ for $i, j = 1, \ldots, n$. We now construct a strictly feasible soultion (X^1, X^2, x^3, x^4) of the dual problem (37). Let $x^3 = (\theta, 0, \ldots, 0) \in \Re^{n+2}$. It clearly satisfies the constraint $-x^3_1 + x^3_2 = -\theta$, and moreover, $x^3 \in \mathrm{ri}(\mathcal{L}^{n+2})$ due to $\theta > 0$. Next, let

$$X^1 = \frac{1}{2(1+\gamma)} \left[\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \\ 1 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \\ 1 \end{pmatrix}^T + \gamma I \right]. \tag{39}$$

In view of this identity and (36), one has $X^1_{22} = 1/2$. Since $\omega > 0$, we know from (13) that $\tilde{c}(\omega) > 0$. This together with (27), (28) and (39), implies that

$$\begin{aligned} -\begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \bullet X^1 &= -\tfrac{1}{2(1+\gamma)} \left[R \bullet \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T + 2h^T \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} + \eta + \gamma \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \bullet I \right] \\ &= -\tfrac{1}{2(1+\gamma)} \left[-\tilde{c}(\omega) + \gamma \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \bullet I \right] \succ 0 \end{aligned}$$

and $X^1 \succ 0$ for sufficiently small positive γ. Now, let

$$X^2_{22} = 2\theta \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X^1_{11}. \tag{40}$$

We next show that $X^2_{22} \succ 0$. Indeed, using (38) and the assumption that $0 \neq F \succeq 0$, we obtain

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot I \succ 0. \tag{41}$$

Further, we have for every $u \in \Re^n$,

$$\begin{aligned} u^T \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot \left[\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T \right] u &= \sum_{i,j} \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bullet (u_i u_j \bar{x}_i \bar{x}_j^T), \\ &= \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bullet \left(\sum_{i,j} u_i u_j \bar{x}_i \bar{x}_j^T \right), \\ &= \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \bullet \left((\sum_i u_i \bar{x}_i)(\sum_i u_i \bar{x}_i)^T \right) \ \geq \ 0, \end{aligned}$$

and hence,

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot \left[\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix}^T \right] \succeq 0.$$

This together with (39)-(41) and the assumption that $\theta > 0$, implies that $X^2_{22} \succ 0$. Letting $X^2_{12} = 0$ and $X^2_{11} = 1$, we immediately see that $X^2 \succ 0$. We also observe that for sufficiently large x^4, (X^1, X^2, x^3, x^4) also strictly satisifes the first constraint of (37). Hence, it is a strictly feasible solution of the dual problem (37). The remaining proof directly follows from strong duality.

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